A multifractal analysis of air temperature signals based on the wavelet leaders method

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Hölder exponent

Definition

Let f be a signal and x_0 a real number. Then f belongs to the Hölder space $C^\alpha(x_0)$ if there exists a polynomial $P_{x_0,\alpha}$ of degree at most α, a positive constant C and a neighborhood V_{x_0} of x_0 satisfying

$$|f(x) - P_{x_0,\alpha}(x)| \leq C|x - x_0|^\alpha$$

for all $x \in V_{x_0}$.

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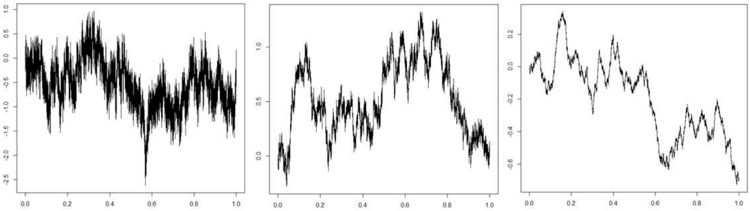
Definition

The Hölder exponent $h(x_0)$ of f at x_0 is defined as the supremum of the exponents α such that f belongs to $C^\alpha(x_0)$:

$$h(x_0) = \sup\{\alpha : f \in C^\alpha(x_0)\}.$$

Monofractality

- Hölder exponent changes from point to point : f multifractal
- Constant Hölder exponent : f monofractal, i.e. f is regularly irregular
- Example of a monofractal function : fractional Brownian motion



Fractional Brownian motions with Hölder exponents 0.2, 0.4, 0.6 almost surely.

Spectrum of singularities

How to characterize the global regularity of a signal ?

Definition

The spectrum of singularities of f is the Hausdorff dimension of the set of points sharing the same Hölder exponent :

$$d_f : h \mapsto \dim_{\mathcal{H}}(\{x_0 \in \mathbb{R} : h(x_0) = h\}),$$

where $\dim_{\mathcal{H}}(X)$ denotes the Hausdorff dimension of the set X.

Corollary : f is monofractal if and only if its spectrum of singularities is reduced to a single point.

Wavelet leaders method (WLM)

1) Wavelet decomposition of the signal :

$$f(x) = \sum_{j,k\in\mathbb{Z}} c_{j,k}\psi(2^j x - k) = \sum_{\lambda\in\Lambda} c_\lambda \psi_\lambda$$

where ψ is a wavelet and $c_{j,k}$ is the wavelet coefficient associated to the dyadic interval λ at scale j and position k :

$$\lambda = \lambda_{j,k} = [2^{-j}k, 2^{-j}(k+1)[$$

and

$$c_{j,k} = 2^j \int_{\mathbb{R}} f(x)\psi(2^j x - k)dx.$$

2) For each λ, compute the wavelet leaders

$$d_\lambda = \sup_{\lambda'\subset\lambda} |c_{\lambda'}|$$

Wavelet leaders method (WLM)

3) Remove the null wavelet leaders and compute

$$S(q,j) = \frac{1}{2^j} \sum_{\lambda \in \Lambda_j} d_\lambda^q,$$

where Λ_j is the set of dyadic intervals at scale j.

4) Compute the function τ defined as

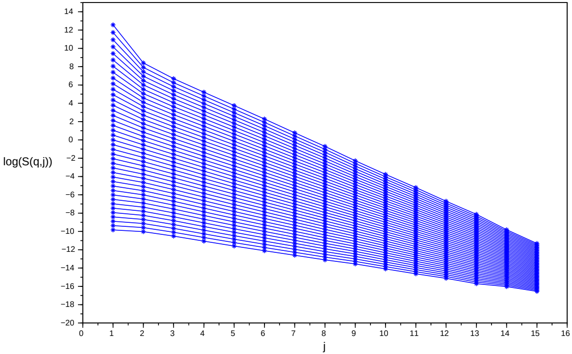
$$\tau(q) = \underline{\lim}_{j \to +\infty} \frac{\log(S(q,j))}{\log 2^{-j}},$$

which is numerically obtained through the slopes of linear regressions at small scales of $\log(S(q,j))$ seen as a function of j.

5) The spectrum of singularities is obtained as

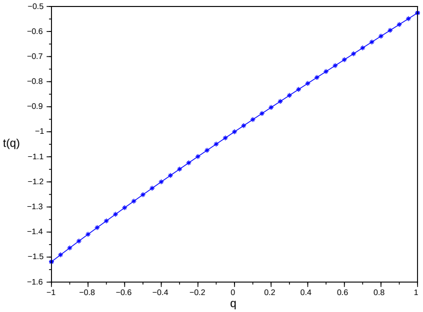
$$d(h) = \inf_q \{qh - \tau(q)\} + 1.$$

Wavelet leaders method (WLM)



$\log(S(q,j))$ for a fractional Brownian motion with Hölder exponent 0.5 with q ranging from -1 to 1.

Wavelet leaders method (WLM)



τ function associated to the previous signal. Linear regression gives a slope of 0.494021.

6) Remark : if τ is a straight line, then f is monofractal, in which case the Hölder exponent of f is the slope of τ.

Wavelet leaders method (WLM)

- Remark : if τ is a straight line, then f is monofractal, in which case the Hölder exponent of f is the slope of τ.
- If f is a monofractal signal with Hölder exponent H, then f belongs to the uniform Hölder space C^H, and a norm in this space is defined by

$$\|f\|_{C^H} = \sup_{j,k}\{|c_{j,k}|/2^{jH}\} := N$$

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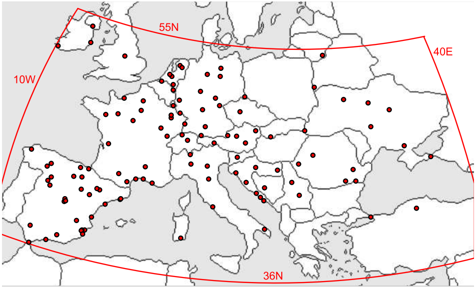
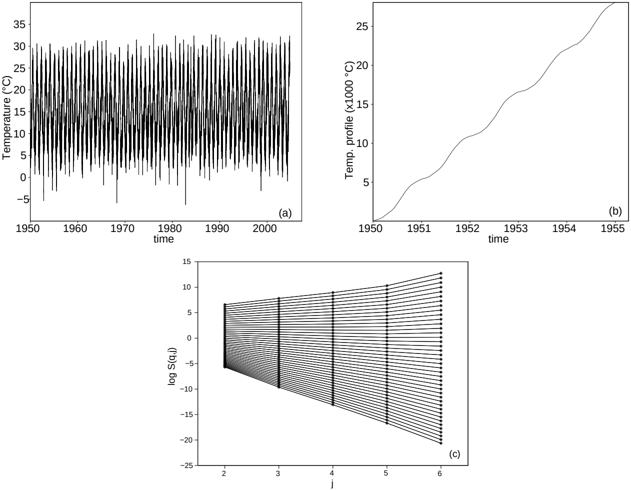
Analyzed data

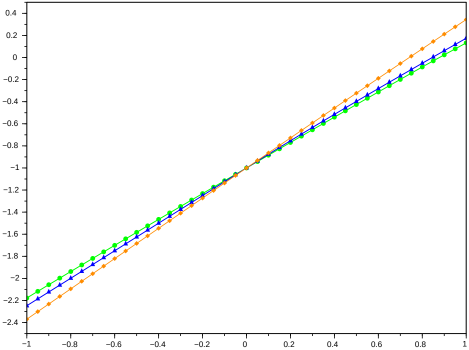


- Daily mean temperature data from 1951 to 2003, calculated as average of minimum and maximum daily temperatures
- Weather stations located below 1000 meters of altitude
- 115 stations selected
- Missing data up to 7%, less than 1% for 97 stations
- Values integrated for more stable numerical results (i.e. x_n replaced by $\sum_{j=1}^{n} x_j$.)

Analyzed data (Granada)



Monofractal nature of the signals

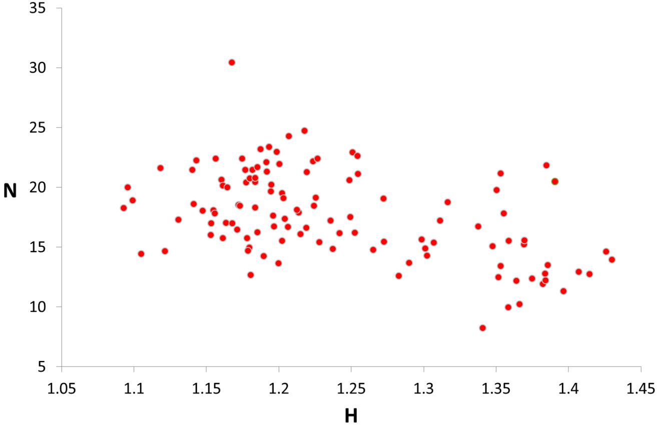


τ functions associated to Aachen (green), Шепетивка (blue) and Granada (orange), with respective slopes 1.156, 1.218, 1.323.

Hölder exponents and norms

- τ linear $\Longrightarrow$ signals are monofractal
- Mean coefficient of determination : $R^2 = 0.9975 \pm 0.0028$
- Hölder exponents ranging from 1.093 to 1.43
- Norms ranging from 8.23 to 30.45

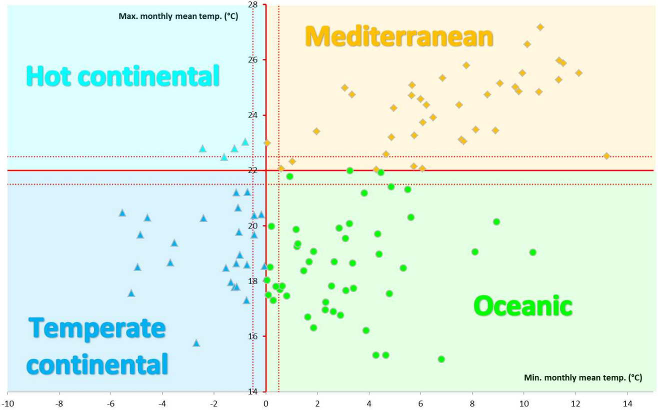
Distribution of the exponents and norms



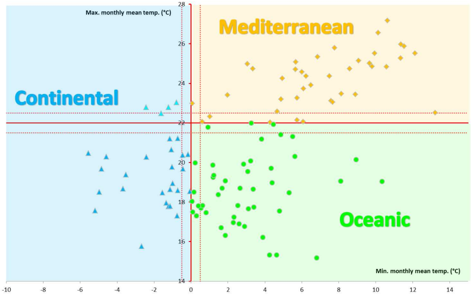
Link with climate types ?

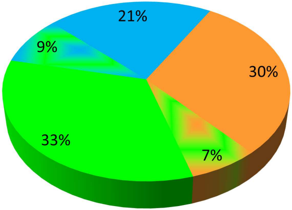
Köppen-Geiger climate classification

Classification based on maximum and minimum monthly mean temperatures (references fixed at 22°C and 0°C). Stations close to 0.5°C of another type of climate were also associated to this second category. Here, precipitations were not taken into account.

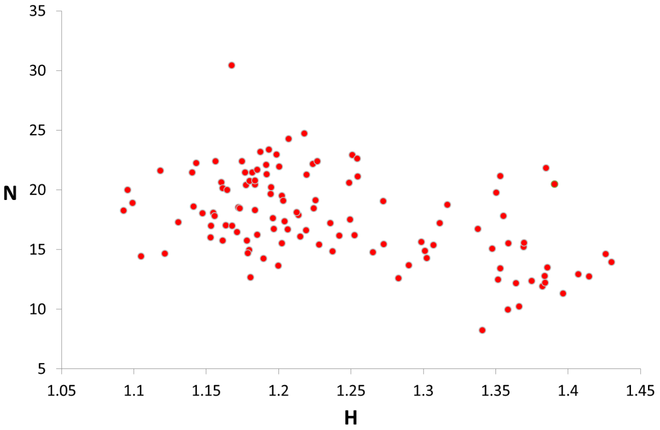


Climate distribution

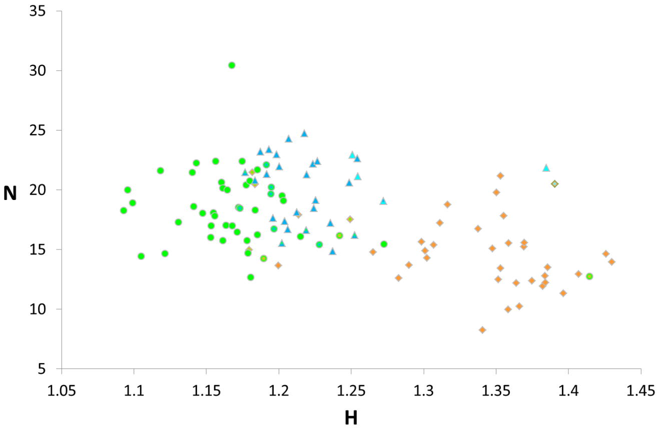




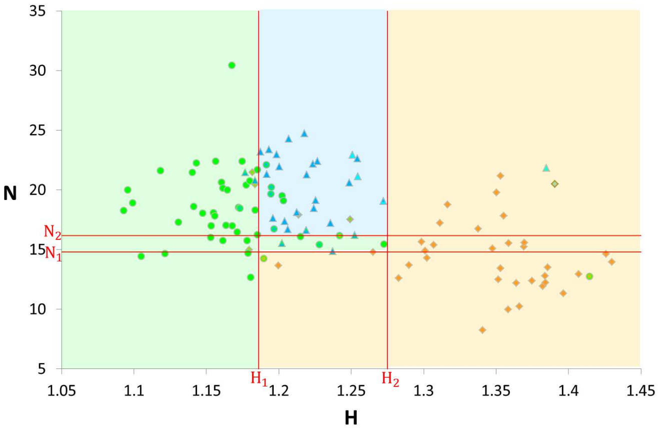
Distribution of the exponents and norms



Distribution of the exponents and norms



Distribution of the exponents and norms



Hölder spaces-based climate classification and results

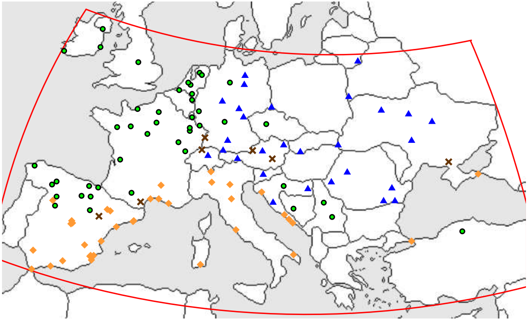
Maximum matching with Köppen-Geiger classification if

$$
\begin{aligned}
H_1 &= 1.186 \\
H_2 &= 1.275 \\
N_1 &= 14.81 \\
N_2 &= 16.18
\end{aligned}
$$

Result : 93.9% correctly associated

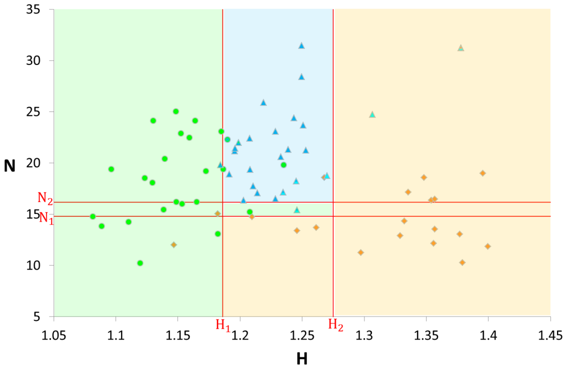
Remark : without the norm, 89.6% correctly associated.

Results on the map



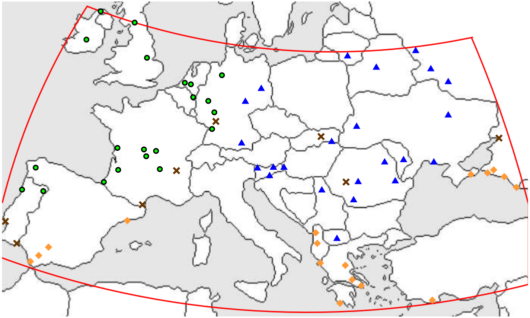
Stations marked with a brown cross are the ones whose type of climate was erroneously predicted. The others were correctly predicted ; green discs stand for Oceanic climate, blue triangles correspond to continental stations and orange orange diamonds are the Mediterranean ones.

Blind test



- 69 other stations
- 40 years of data between 1951 and 2003

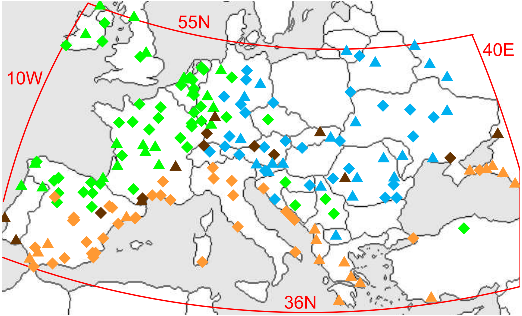
Blind test



Result : 88.4% correctly associated

Remark : without the norm, 84.1% correctly associated.

All the stations



115 stations of reference (diamonds) + 69 stations of the blind test (triangles)

Overall result: 91.8% correctly associated

Discussion of the results

Results

Oceanic stations	$\longleftrightarrow$	Lowest Hölder exponents
Continental stations	$\longleftrightarrow$	Intermediate Hölder exponents
Mediterranean stations	$\longleftrightarrow$	Largest Hölder exponents

Discussion

- On a daily basis, Oceanic climate is more irregular than the Continental weather, which is less regular than Mediterranean climate.
- Explanation could be the North Atlantic Oscillation (NAO), anticyclonic conditions in Southern Europe, ...

Conclusions and future work

Conclusions

- WLM shows surface air temperatures signals are monofractal signals
- Their belonging to functional spaces reflects their temperature-based Köppen-Geiger climate type
- Algorithm and results confirmed through blind tests

Future work

- Checking of the validity of current climatic models
- Analysis of other climate indices (pressure, precipitation,...)
- Generalization to global temperatures

Some references

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