A whistle-stop tutorial on observational turbulence studies

Background and motivation for a statistical treatment

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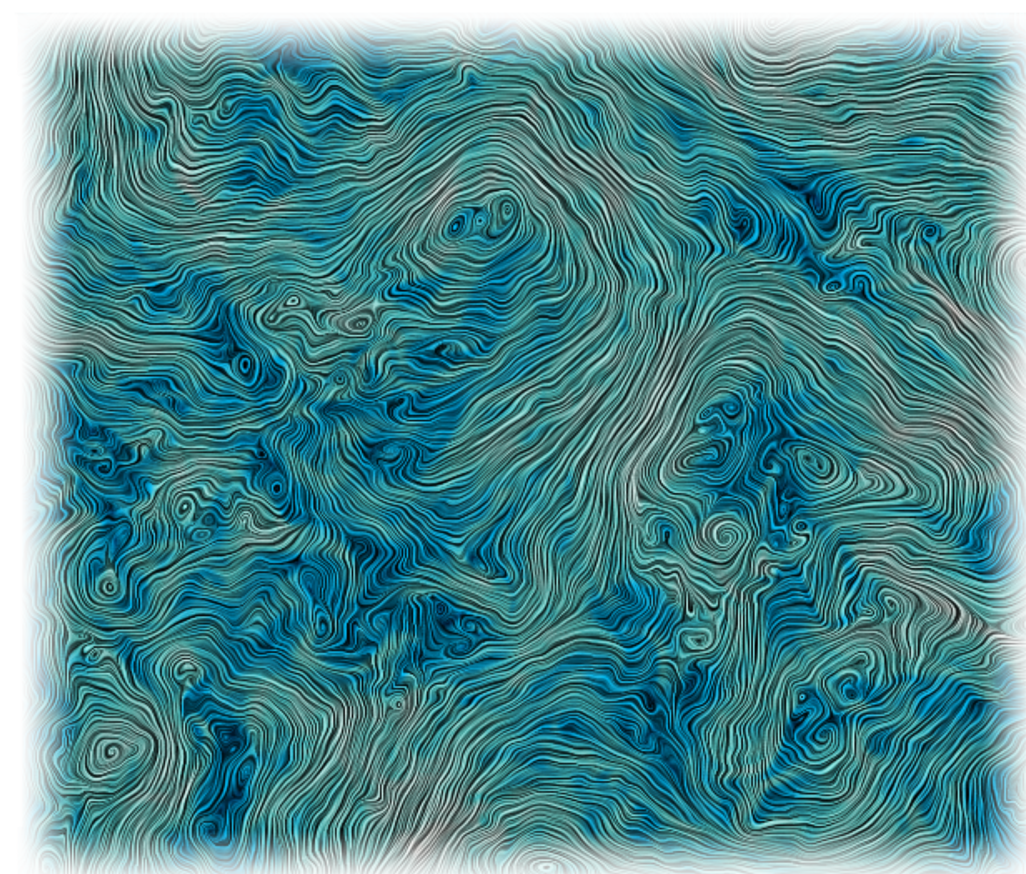
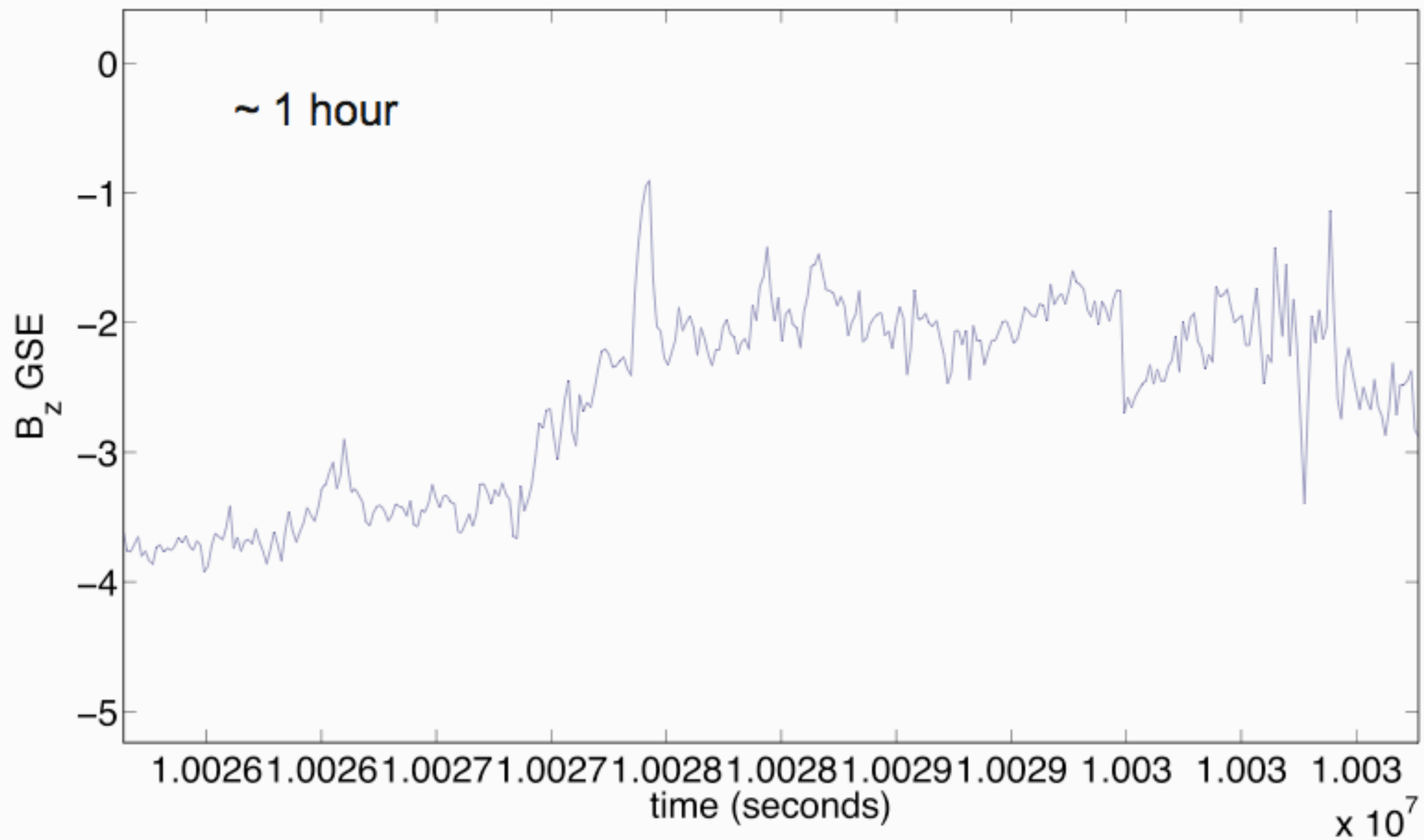
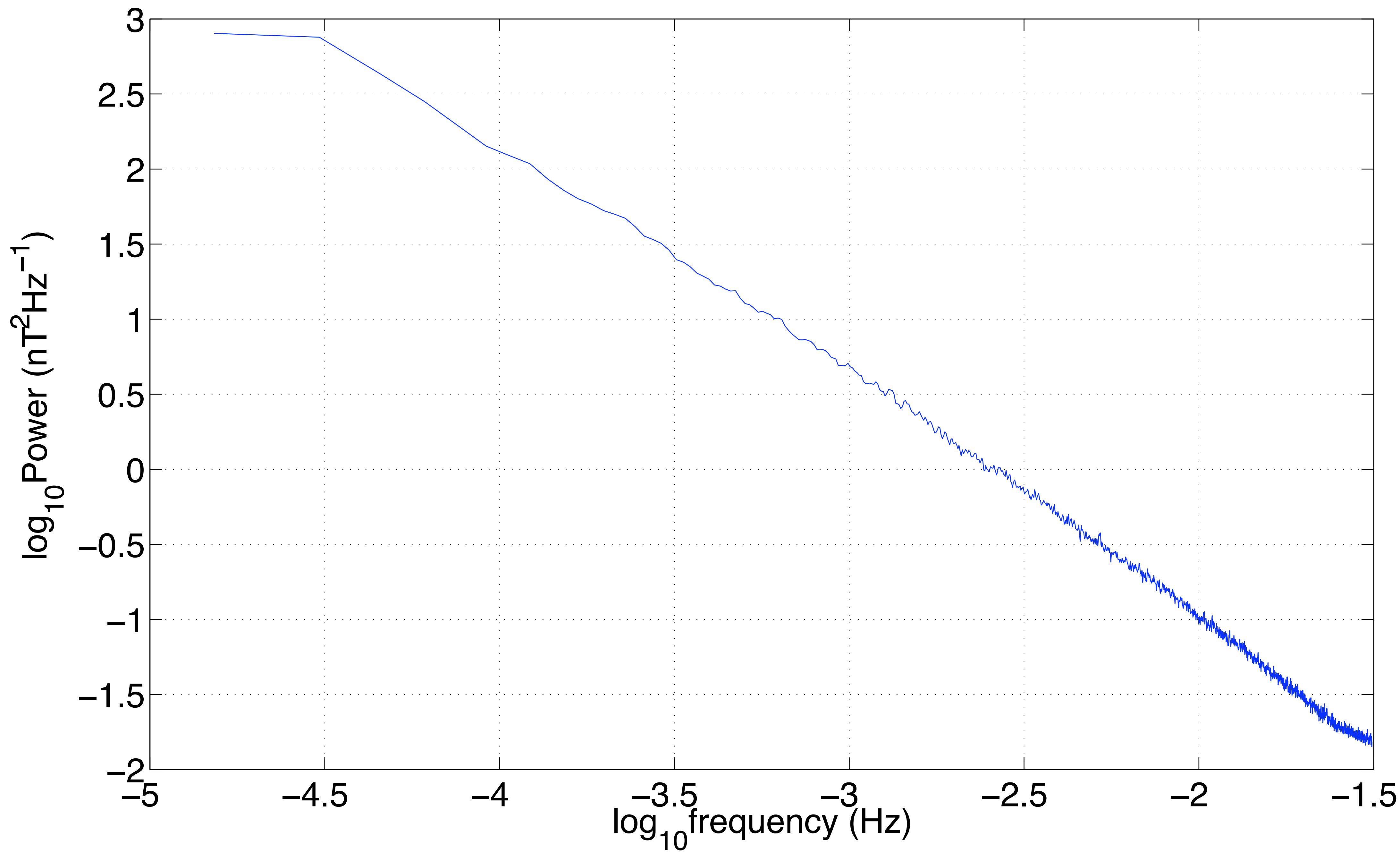


motivation



~ 1 hour
0
−1
−2
−3
−4
−5
B_z GSE
1.0026
1.0026
1.0027
1.0027
1.0028
1.0028
1.0029
1.0029
1.003
1.003
1.003
time (seconds)
x 10^7

motivation



3
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
log10Power (nT2Hz−1)
−5
−4.5
−4
−3.5
−3
−2.5
−2
−1.5
log10frequency (Hz)

outline


What do you think?
АКАДЕМИК
КОЛМОГОРОВ
АНДРЕЙ НИКОЛАЕВИЧ
1903-1987
1912
1987

outline


Why, energy transfer & scaling of course, you silly man!
АКАДЕМИК
КОЛМОГОРОВ
АНДРЕЙ НИКОЛАЕВИЧ
1903-1987

outline

- Equations of motion and phenomenology of energy transfer in r and k-space for iHI turbulence.
- Richardson energy cascade and the 5/3rd energy spectrum (power spectral density)
- Measurement and ensembles
- Higher order two-point statistics
- Some real data from the solar wind
- 4/5th (third order) law
- fractal models (if time permits)

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statistical theory of turbulence

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

Incompressible fluid Navier-Stokes equations

statistical theory of turbulence

$$\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla' \mathbf{u}' = -\frac{1}{\rho} \nabla p' + \frac{1}{Re} \nabla'^2 \mathbf{u}'$$

$$\nabla' \cdot \mathbf{u}' = 0$$

$$Re = \frac{LV}{\nu}$$

Reynolds number

Dimensionless Navier-Stokes equations

the turbulence problem (amongst others)

To understand better the phenomenology, and thus dynamics, of turbulence

statistical theory of turbulence

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}$$

$$\mathbf{u} = \bar{U}_i + \tilde{u}_i$$

$$\nabla \cdot \mathbf{u} = 0$$

Reynolds decomposition

statistical theory of turbulence (RANS)

$$\frac{\partial \bar{U}_i}{\partial t} + \frac{\partial}{\partial x_j}\bar{U}_i\bar{U}_j = -\frac{1}{\rho}\partial_i \bar{P} + \nu \partial^2 \bar{U}_i$$

$$+\frac{\partial}{\partial x_j}\langle \tilde{u}_i \tilde{u}_j \rangle$$

Reynolds stress tensor

(two point spatial correlation/covariance)

(will stay with 2-point statistics)

statistical theory of turbulence

$$\langle \tilde{u}_i \tilde{u}_j \rangle \overset{FT}{\longleftrightarrow} P(k)$$

Wiener-Khinchin theorem

energy transfer in real space (Karman-Howarth eq)

$$\frac{\partial}{\partial t}\left\langle u_i u'_k \right\rangle + \frac{\partial}{\partial x'_j}\left\langle u_i u_j u_k \right\rangle + \frac{\partial}{\partial x'_j}\left\langle u_i u'_j u'_k \right\rangle$$
$$= -\frac{1}{\rho}\left\langle u'_k \frac{\partial p}{\partial x_i} + u_i \frac{\partial p'}{\partial x_k} \right\rangle + \nu \left\langle u_i \nabla'^2 u'_k + u'_k \nabla^2 u_i \right\rangle$$

energy transfer in real space (Karman-Howarth eq)

$$\frac{\partial}{\partial t}\left\langle u_i u'_k\right\rangle + \frac{\partial}{\partial x'_j}\left\langle u_i u_j u_k\right\rangle + \frac{\partial}{\partial x'_j}\left\langle u_i u'_j u'_k\right\rangle$$
$$= -\frac{1}{\rho}\left\langle u'_k \frac{\partial p}{\partial x_i} + u_i \frac{\partial p'}{\partial x_k}\right\rangle + \nu \left\langle u_i \nabla'^2 u'_k + u'_k \nabla^2 u_i\right\rangle$$

$$\frac{2}{3}\frac{\partial E}{\partial t} = -\frac{2}{3}\varepsilon = \frac{1}{2}\frac{\partial S_2}{\partial t} + \frac{1}{6r^4}\frac{\partial}{\partial r}(r^4 S_3) - \frac{\nu}{r^4}\frac{\partial}{\partial r}\left(r^4 \frac{\partial S_2}{\partial r}\right)$$

energy transfer in real space (Karman-Howarth eq)

$$\frac{\partial}{\partial t}\langle u_i u'_k\rangle + \frac{\partial}{\partial x'_j}\langle u_i u_j u_k\rangle + \frac{\partial}{\partial x'_j}\langle u_i u'_j u'_k\rangle$$

$$= -\frac{1}{\rho}\langle u'_k \frac{\partial p}{\partial x_i} + u_i \frac{\partial p'}{\partial x_k}\rangle + \nu\langle u_i \nabla'^2 u'_k + u'_k \nabla^2 u_i\rangle$$

$$\frac{2}{3}\frac{\partial E}{\partial t} = -\frac{2}{3}\varepsilon = \frac{1}{2}\frac{\partial S_2}{\partial t} + \frac{1}{6r^4}\frac{\partial}{\partial r}(r^4 S_3) - \frac{\nu}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial S_2}{\partial r}\right)$$

the closure problem

Fourier (k-space) -- mode-coupling

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] u_\alpha(\mathbf{k}, t) = M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3\mathbf{j} u_\beta(\mathbf{j}, t) u_\gamma(\mathbf{k} - \mathbf{j}, t) + f_\alpha(\mathbf{k}, t)$$

$$k_\alpha u_\alpha(\mathbf{k}, t) = 0$$

$$\frac{\partial}{\partial t}\mathbf{u}(\mathbf{x}, t) - \nu \nabla^2 \mathbf{u}(\mathbf{x}, t) = -\mathbf{u}(\mathbf{x}, t) \cdot \nabla \mathbf{u}(\mathbf{x}, t) - \frac{1}{\rho}\nabla p(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t)$$

$$\nabla \cdot \mathbf{u}(\mathbf{x}, t) = 0$$

energy transfer in Fourier space (Lin equation)

$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right] \langle u_{-k} u_k \rangle = M_k \left\langle u_{-k} u_j u_{k-j} \right\rangle$$
$$+ M_{-k} \left\langle u_k u_{-j} u_{-k+j} \right\rangle + \langle u_{-k} f_k \rangle + \langle f_{-k} u_k \rangle$$

energy transfer in Fourier space (Lin equation)

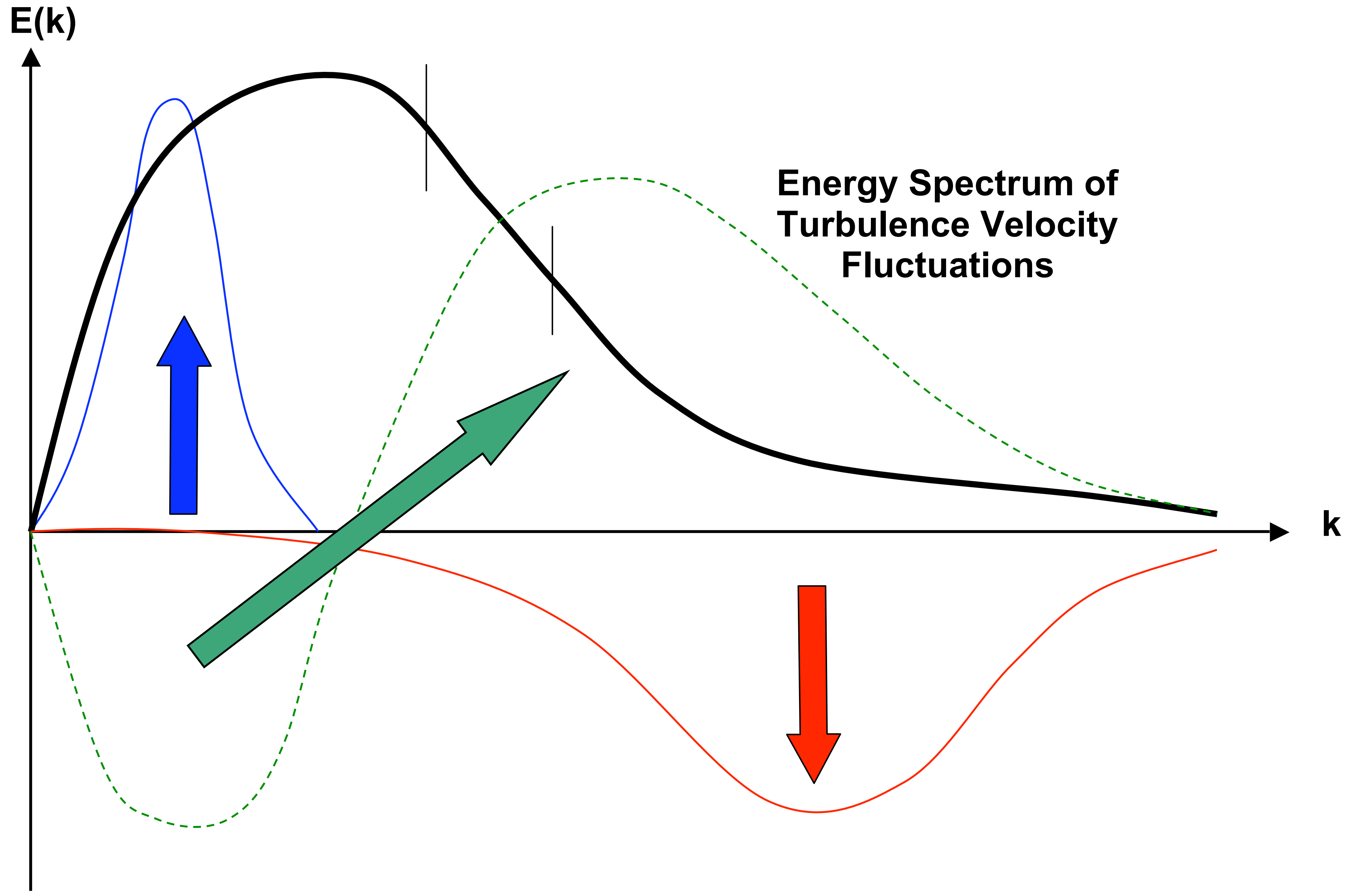
$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right] \langle u_{-k} u_k \rangle = M_k \left\langle u_{-k} u_j u_{k-j} \right\rangle$$
$$+ M_{-k} \left\langle u_k u_{-j} u_{-k+j} \right\rangle + \langle u_{-k} f_k \rangle + \langle f_{-k} u_k \rangle$$

$$T(k,t) \;=\; M_k \left\{ \left\langle u_{-k} u_j u_{k-j} \right\rangle - \left\langle u_k u_{-j} u_{-k+j} \right\rangle \right\}$$

$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right] E(k,t) = T(k,t) + W(k,t)$$

energy transfer in Fourier space (Lin equation)

$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right] E(k,t) = T(k,t) + W(k,t)$$



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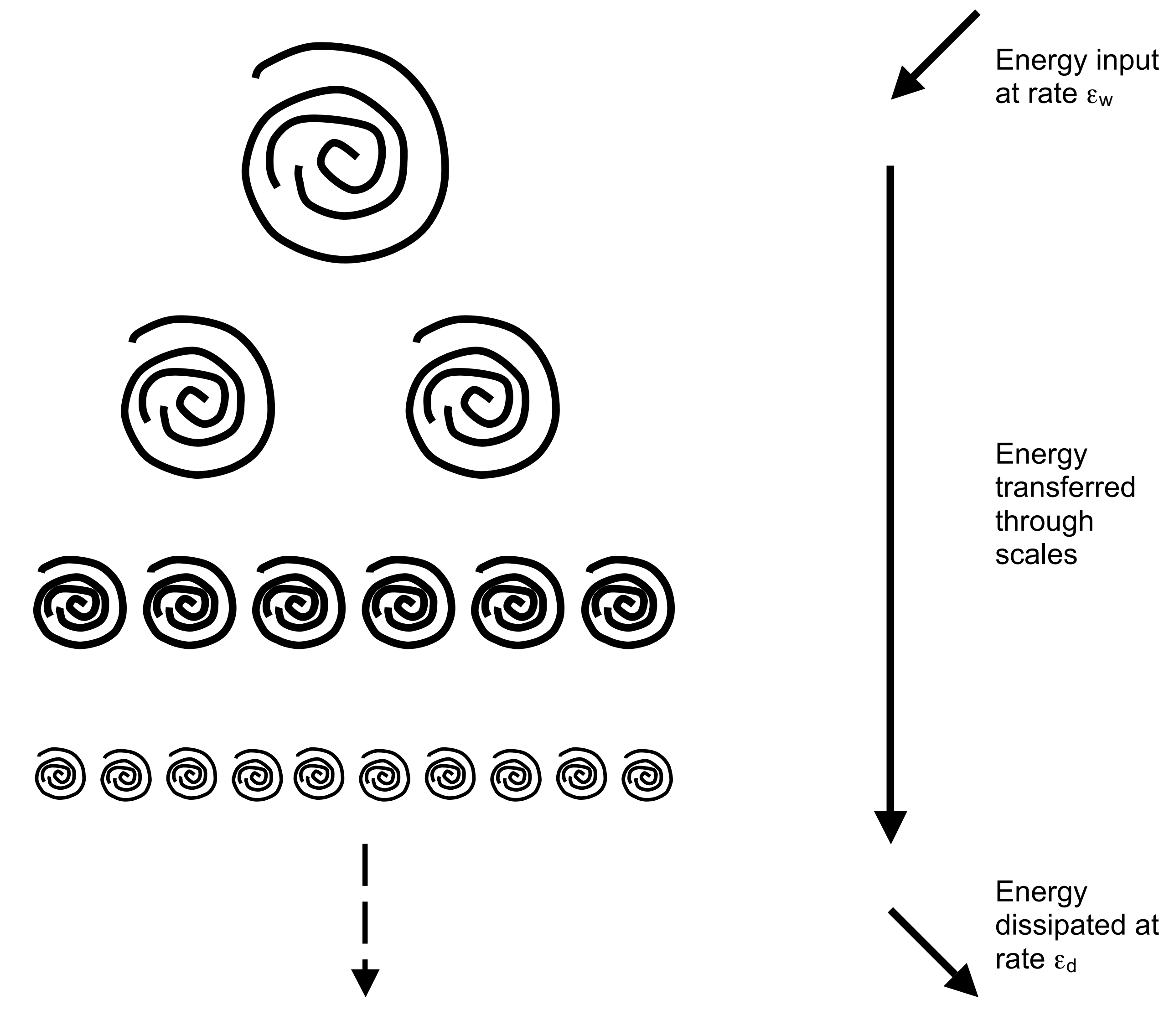
Kolmogorov scaling and Richardson cascade



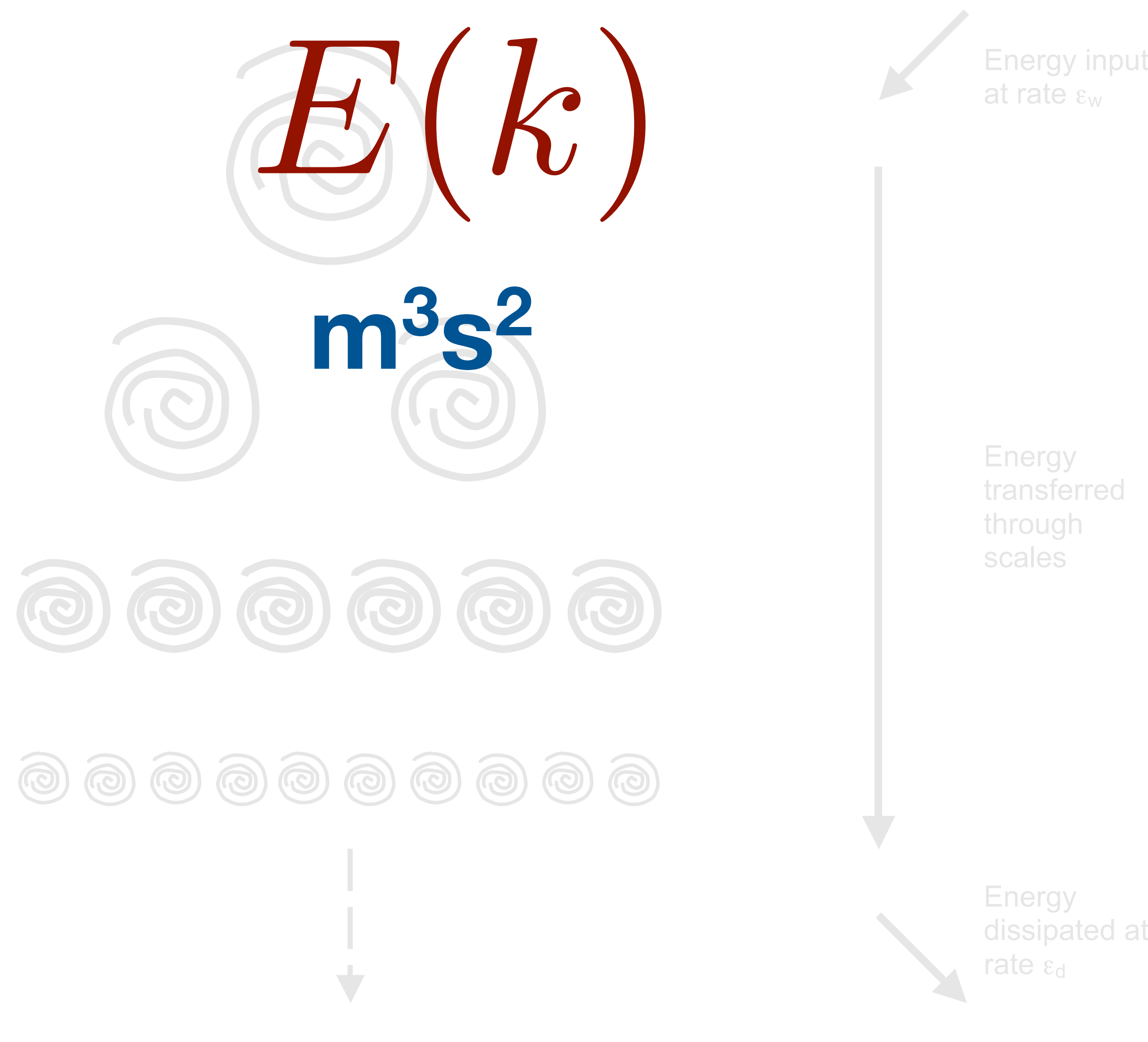
Kolmogorov scaling and Richardson cascade

ε

$\mathbf{m^2 s^3}$

$E(k)$

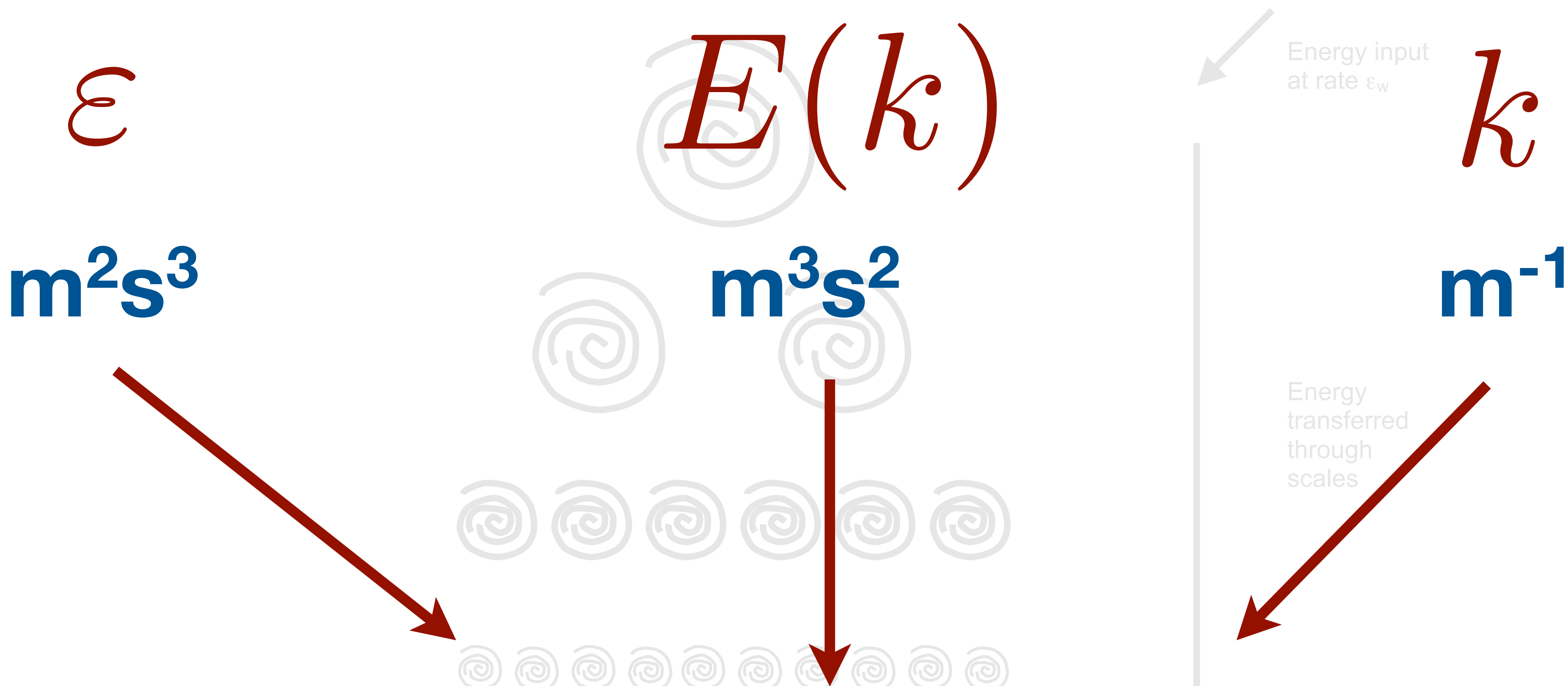
$\mathbf{m^3 s^2}$



k

$\mathbf{m^{-1}}$

Kolmogorov scaling and Richardson cascade



$$E(k) \sim \varepsilon^{2/3} k^{-5/3}$$

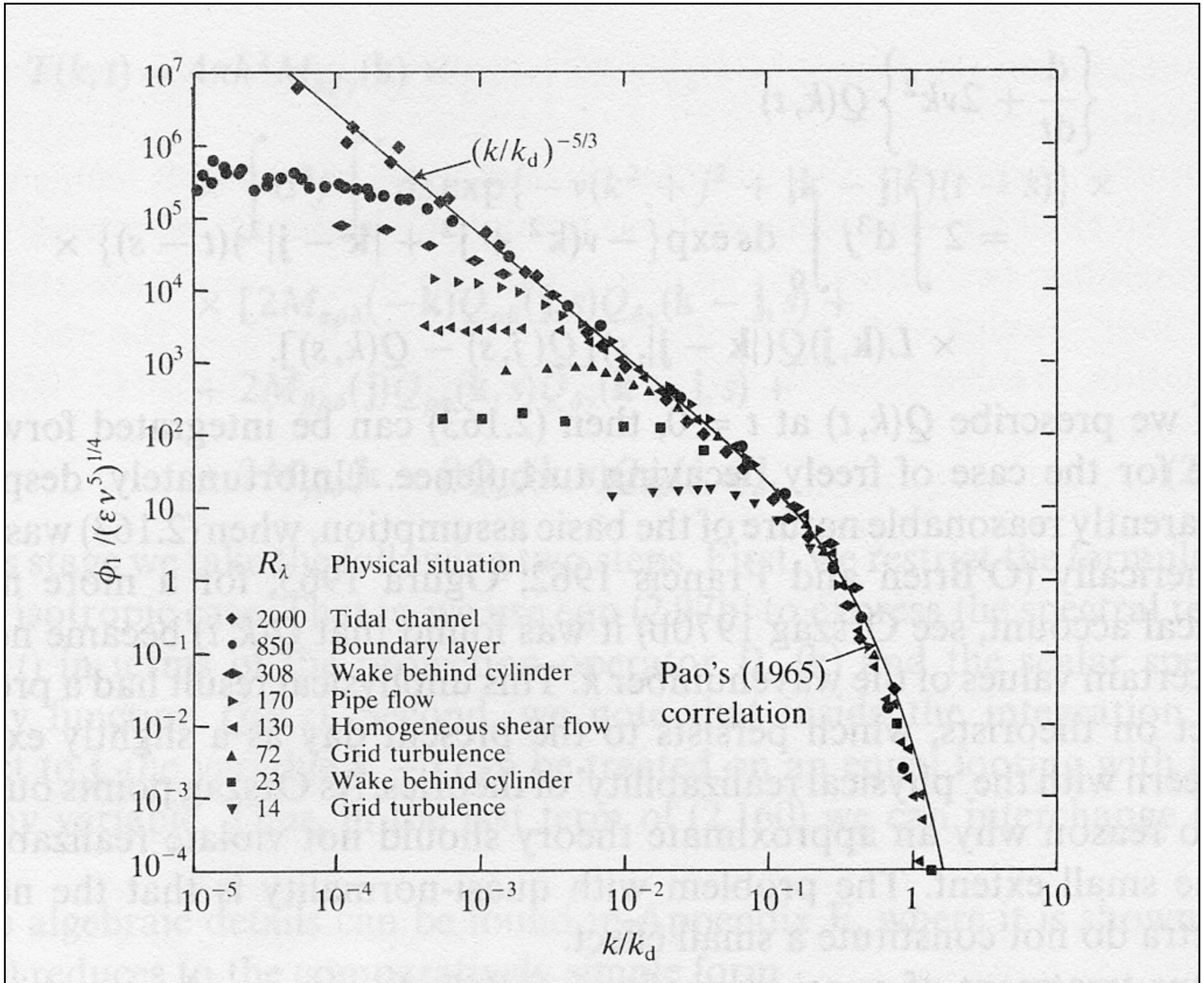
more dimensionless numbers

$$k_d = \left(\frac{\varepsilon}{\nu^3} \right)^{1/4}$$

$$l_d = 1/k_d$$

$$Re = (L/l_d)^{4/3}$$

Kolmogorov scaling and Richardson cascade



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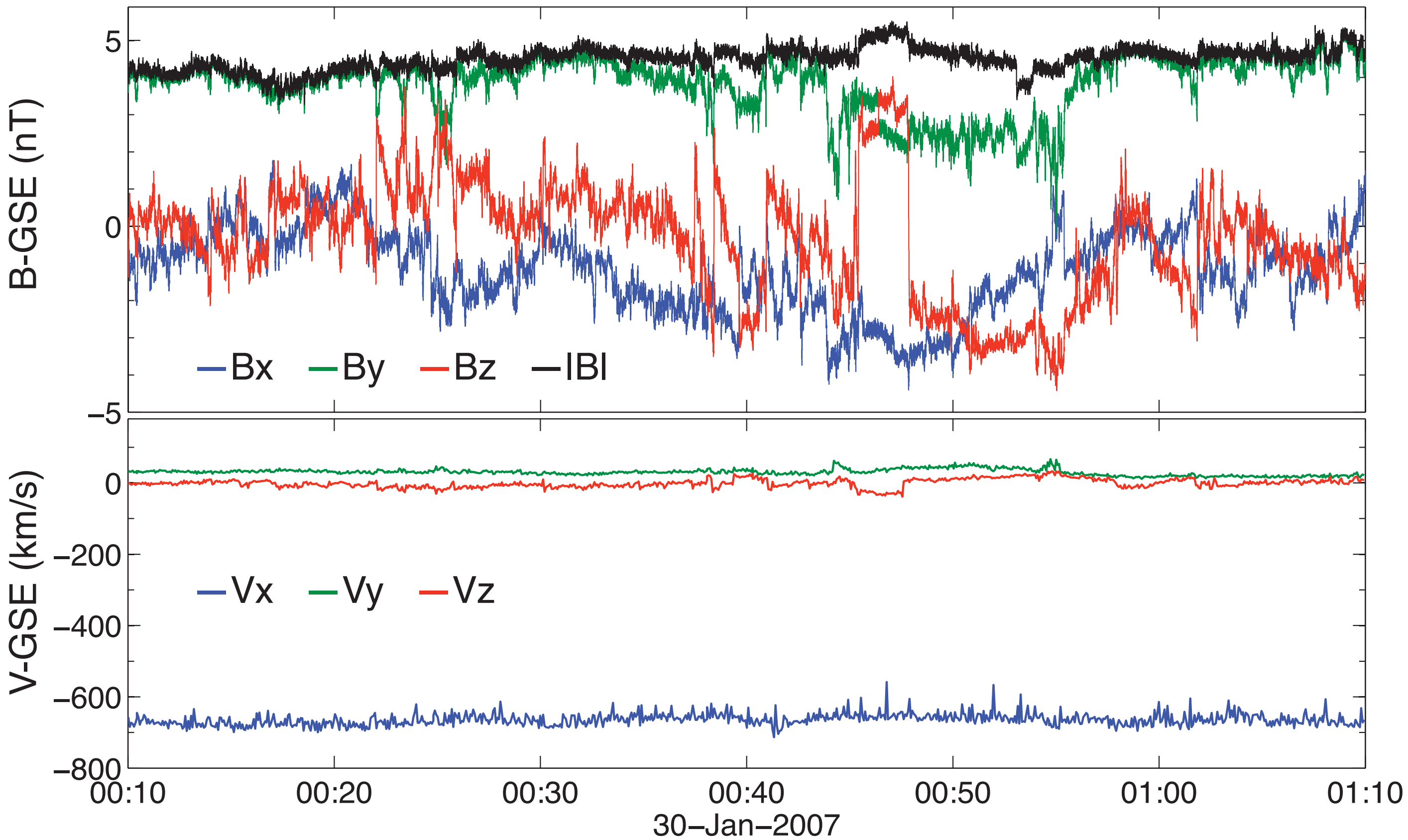
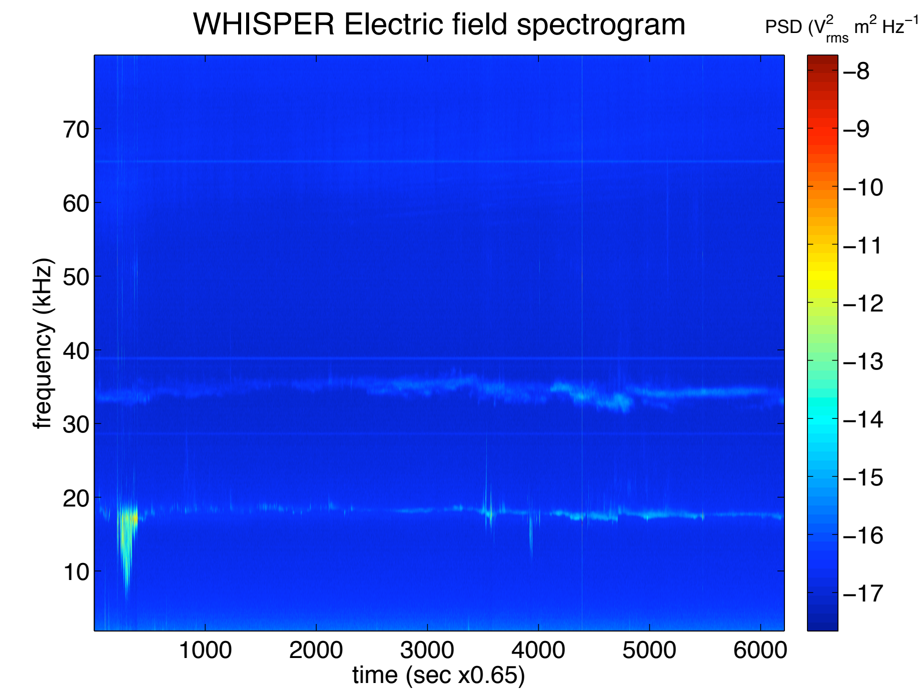
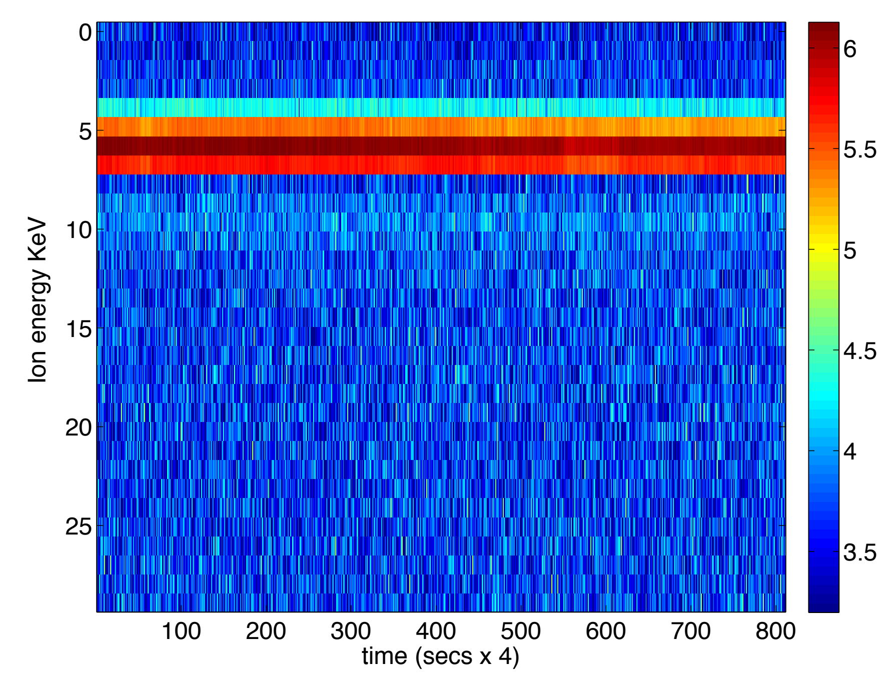
ensembles, ergodicity & statistical stationarity







$n_e \sim 4\ cm^{-3}$	$V_A \sim 50\ km\ s^{-1}$	$Ti_{\perp} \sim 24\ eV$
$\beta_p \sim 1$	$\|B\| \sim 4\ nT$	$Te \sim 22\ eV$
$\rho i \sim 111\ km$	$\rho e \sim 2\ km$	$Tp \sim 15\ eV$

stationary plasma parameters

Frozen turbulence & Taylor's hypothesis

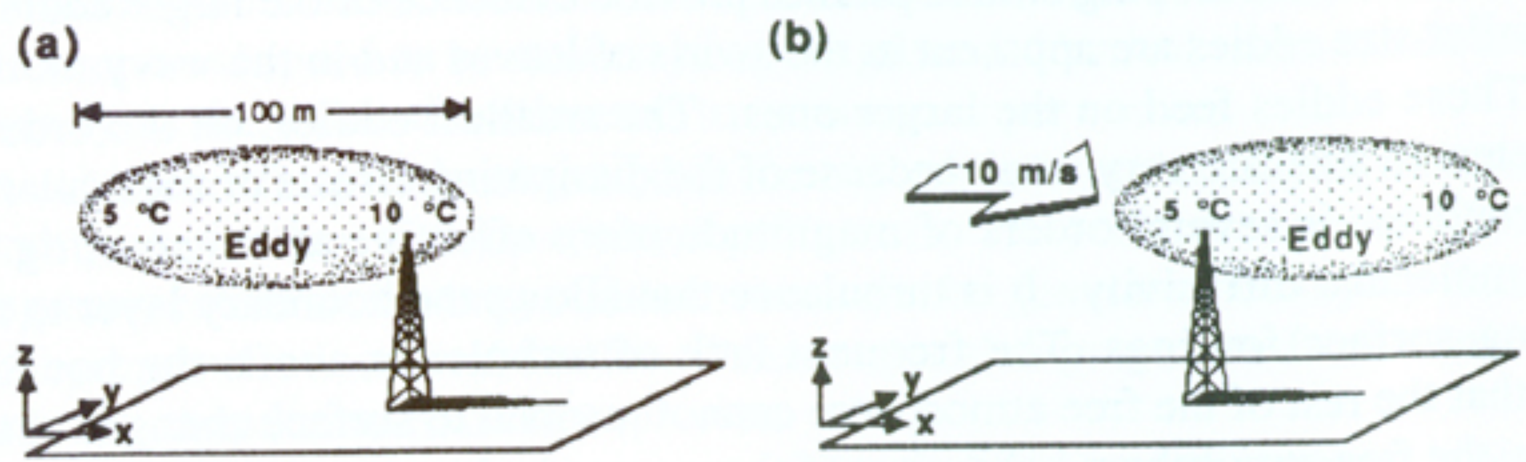


Fig. 1.4 Illustration of Taylor's hypothesis. (a) An eddy that is 100 m in diameter has a 5 ° C temperature difference across it. (b) The same eddy 10 seconds later is blown downwind at a wind speed of 10 m/s.

from Stull 1988

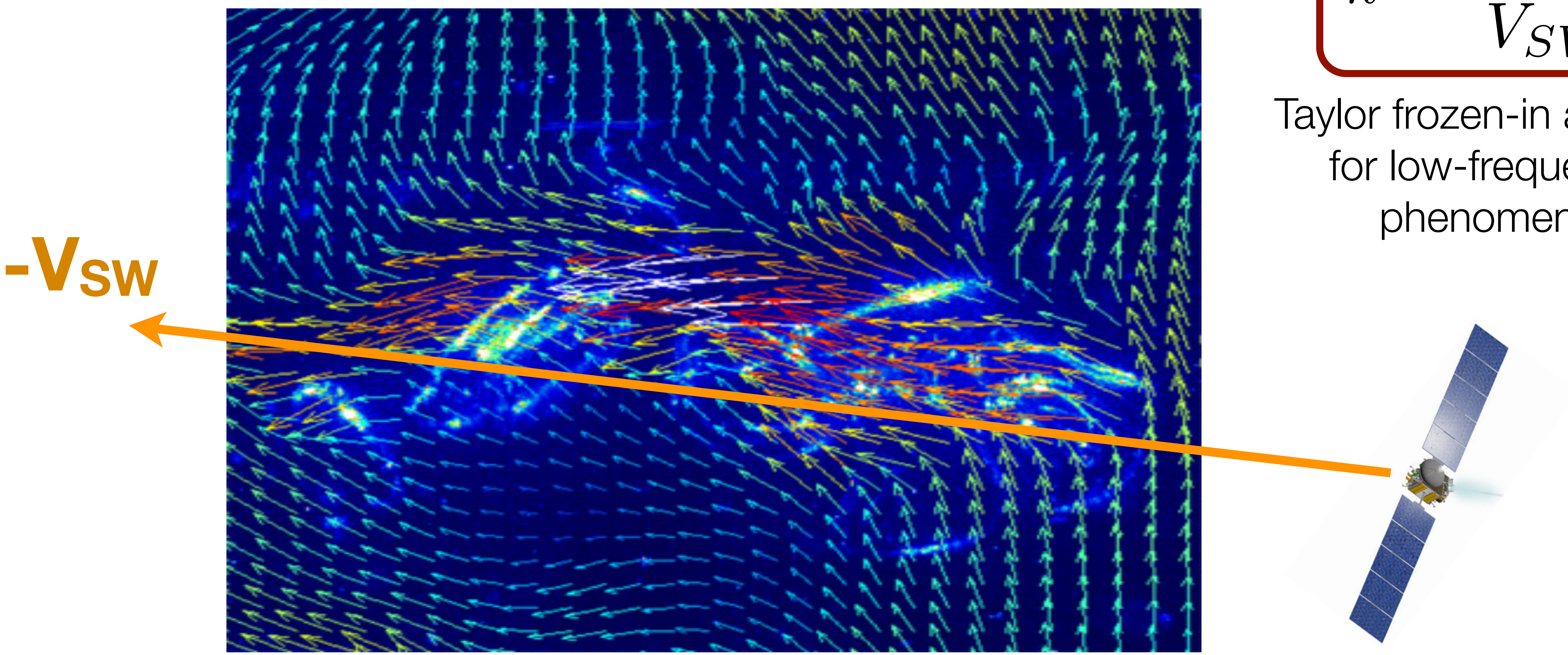
$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{V} \cdot \nabla \mathbf{u}$$

$$\sim \mathbf{k} \cdot \mathbf{V}$$

angles of measurement w.r.t. B

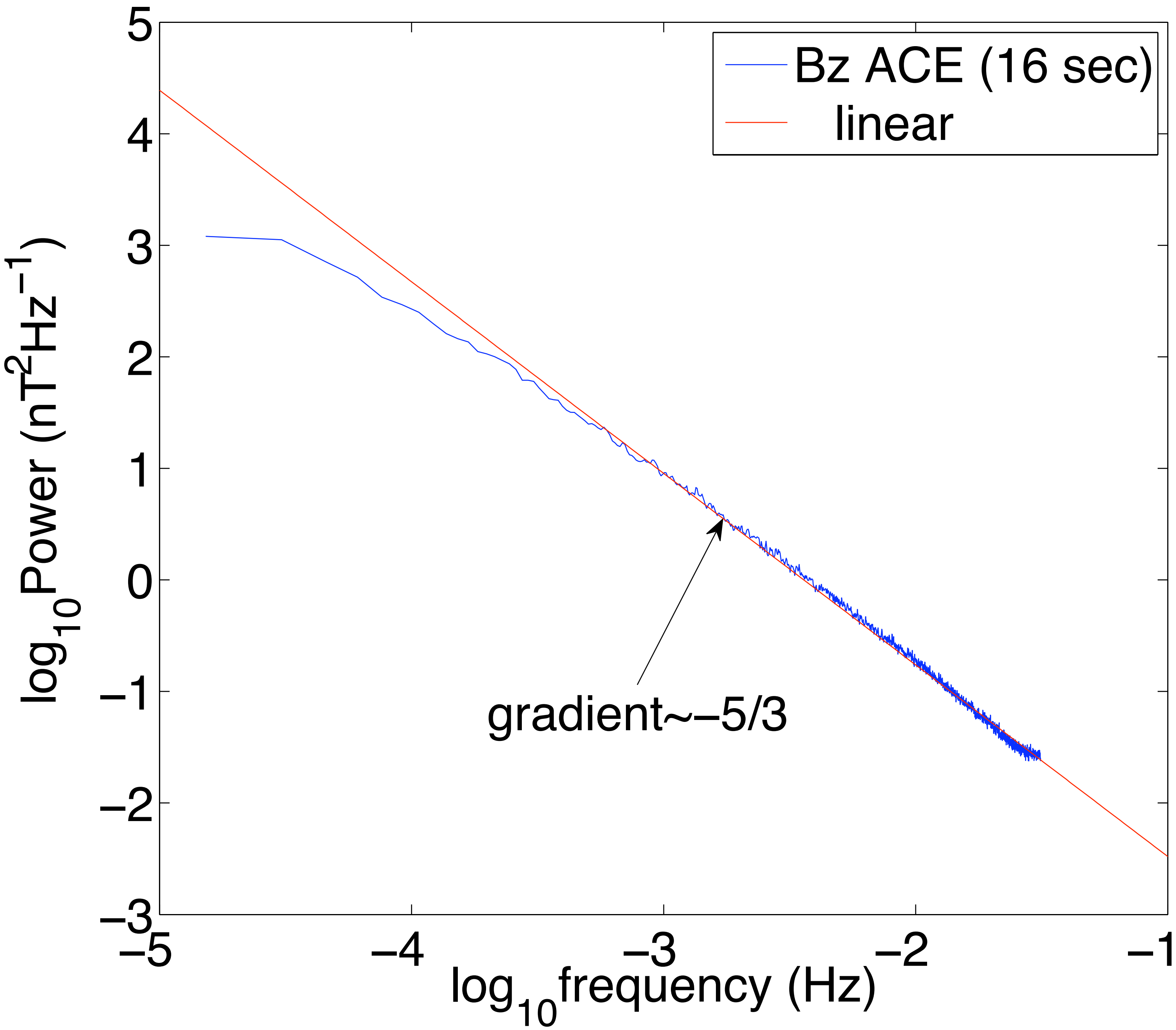
$$k = \frac{2\pi f_{sc}}{V_{SW}}$$

Taylor frozen-in approx, for low-frequency phenomena

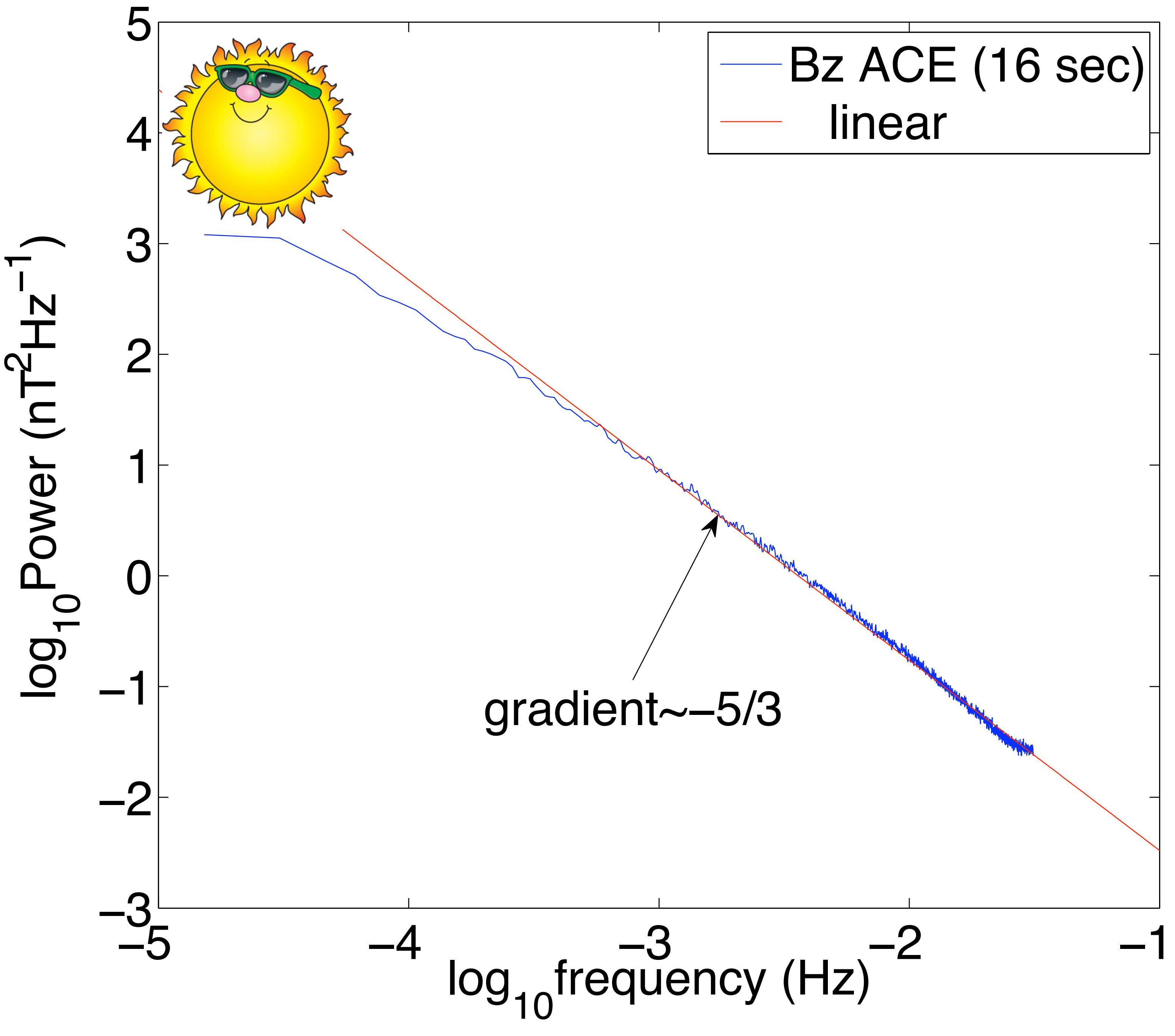


MHD $\frac{V_A}{V_{SW}} < 1$ else $\frac{V_\phi}{V_{SW}} < 1$

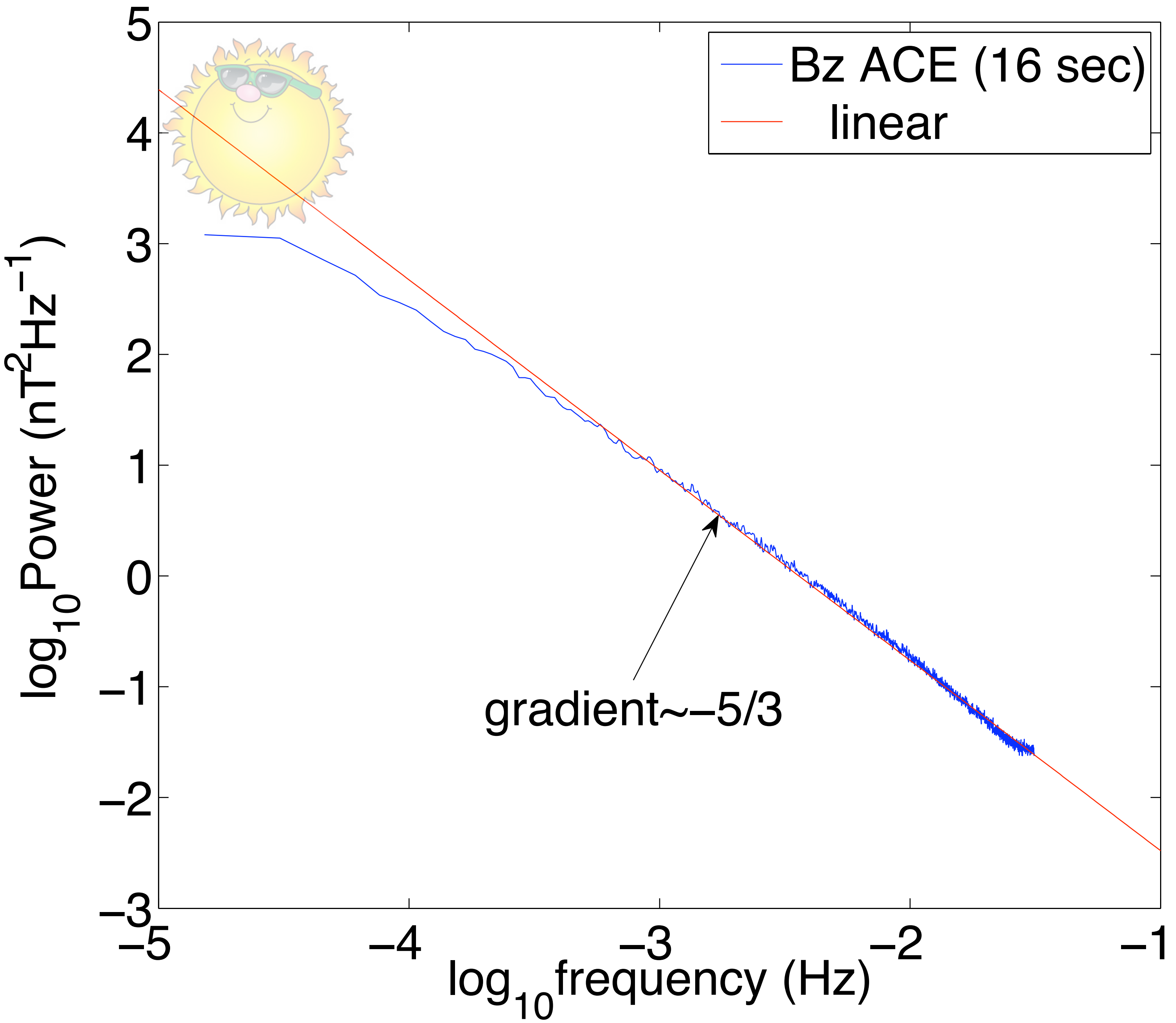
MHD inertial range (spectral) scaling exponents

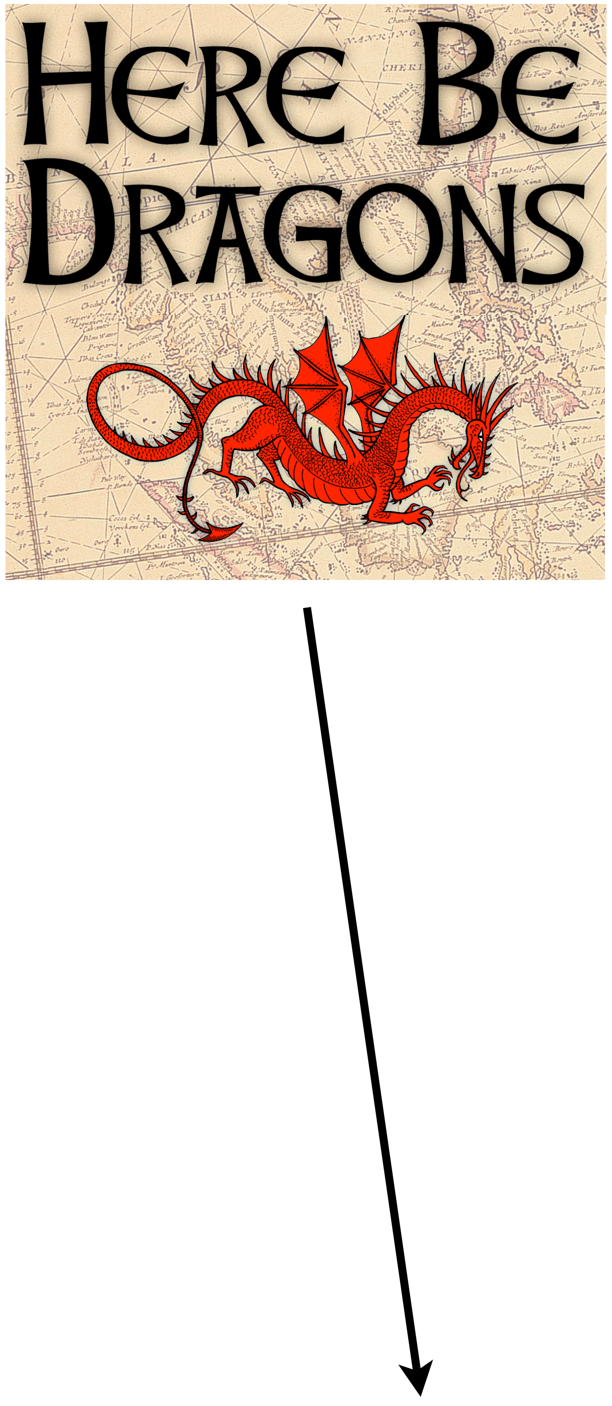


MHD inertial range (spectral) scaling exponents

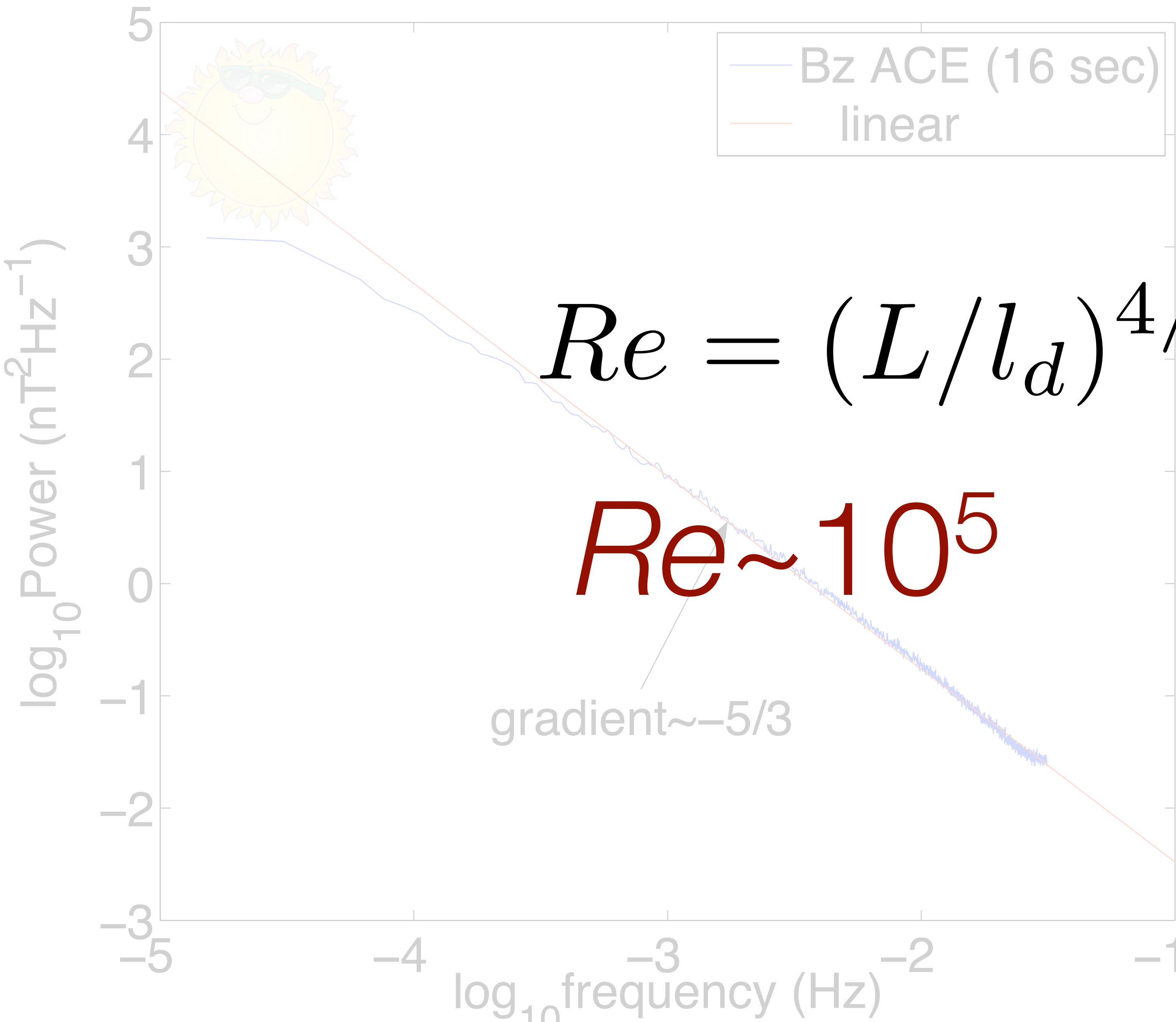


MHD inertial range (spectral) scaling exponents





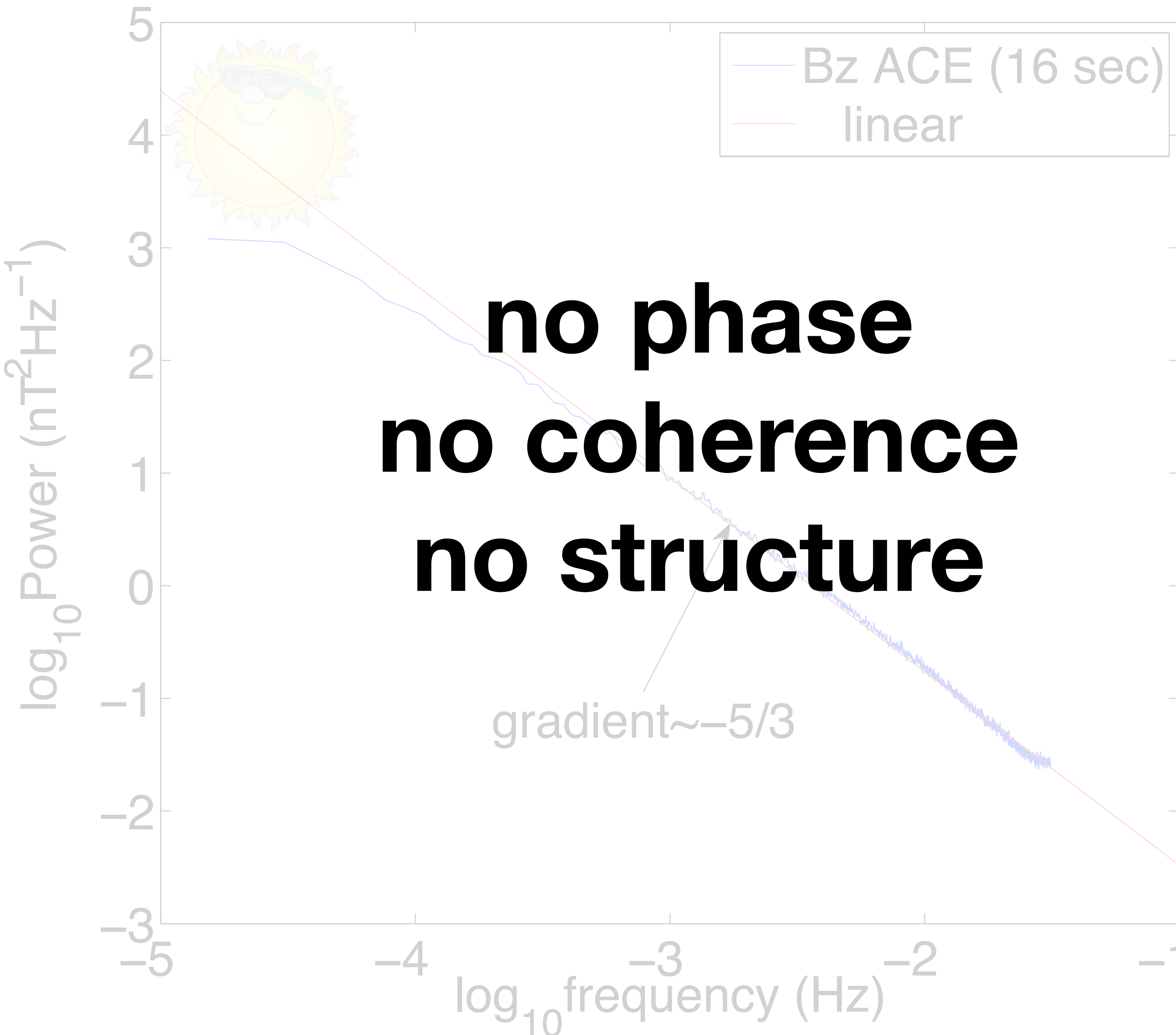
MHD inertial range (spectral) scaling exponents



$$Re = (L/l_d)^{4/3}$$

$Re \sim 10^5$

MHD inertial range (spectral) scaling exponents



no phase
no coherence
no structure

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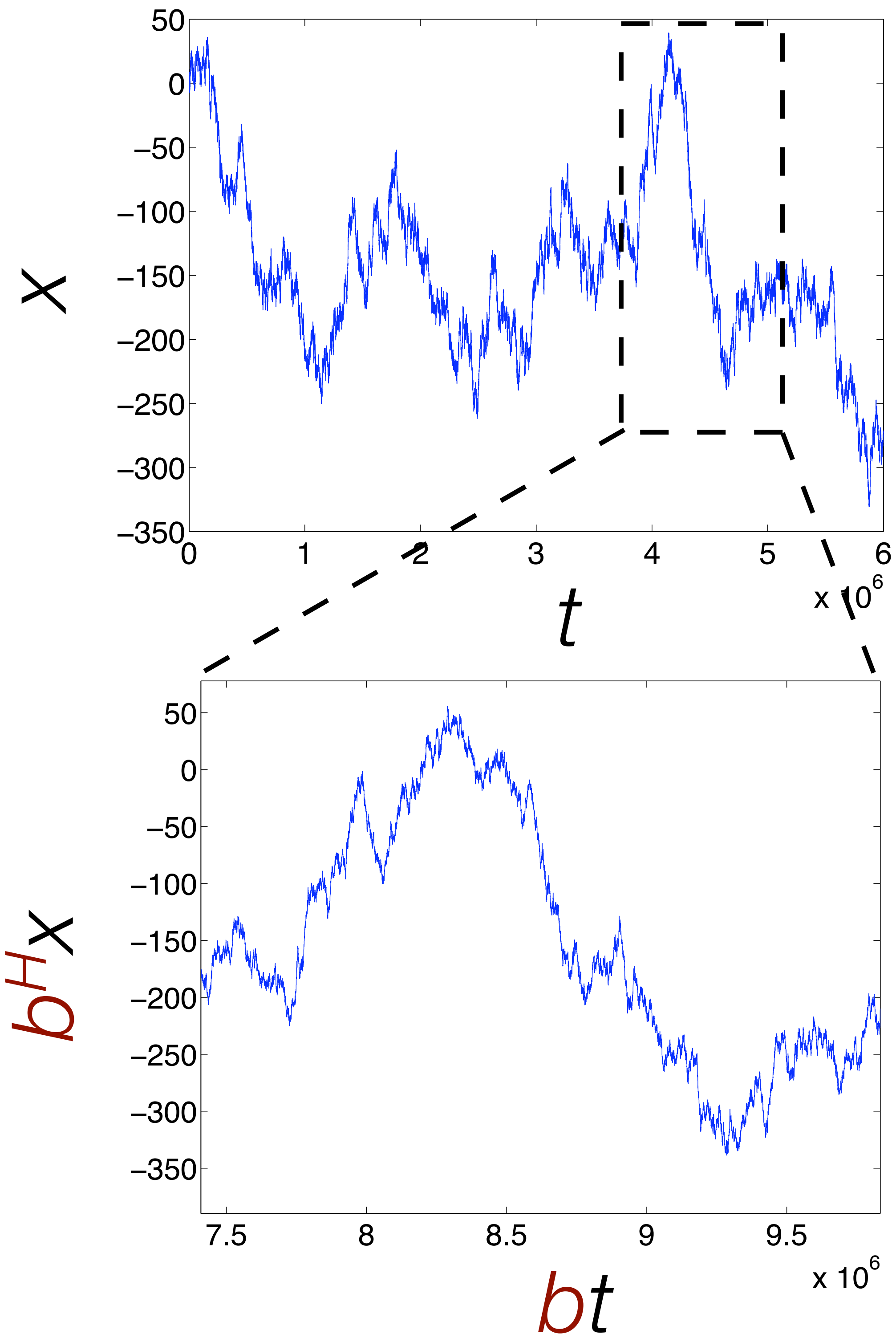
Scaling

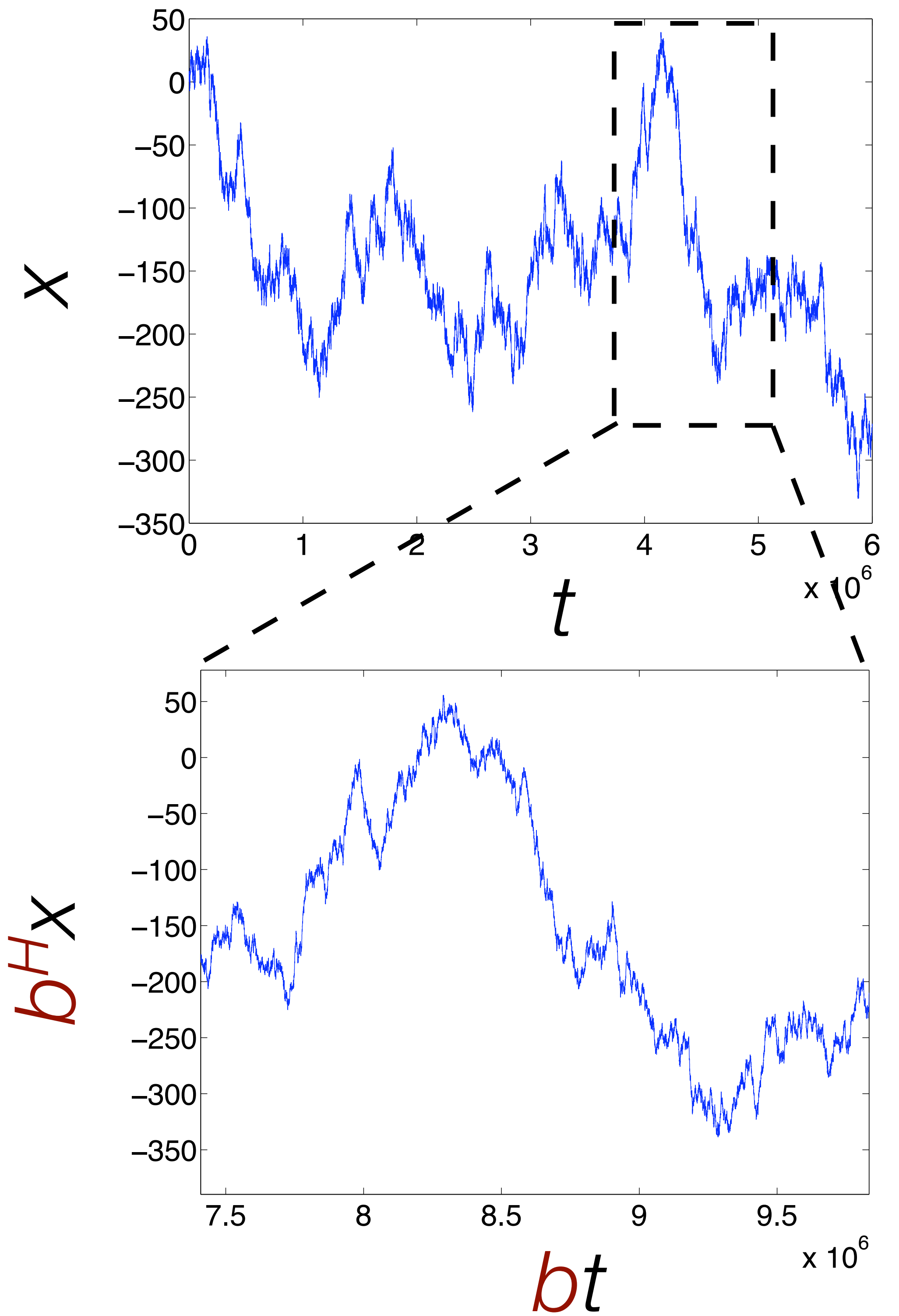
(beyond spectra)

scaling, fractals and all that jazz ...



Change scale from t to bt
AND
scale x to $b^H x$

scaling, fractals and all that jazz ...

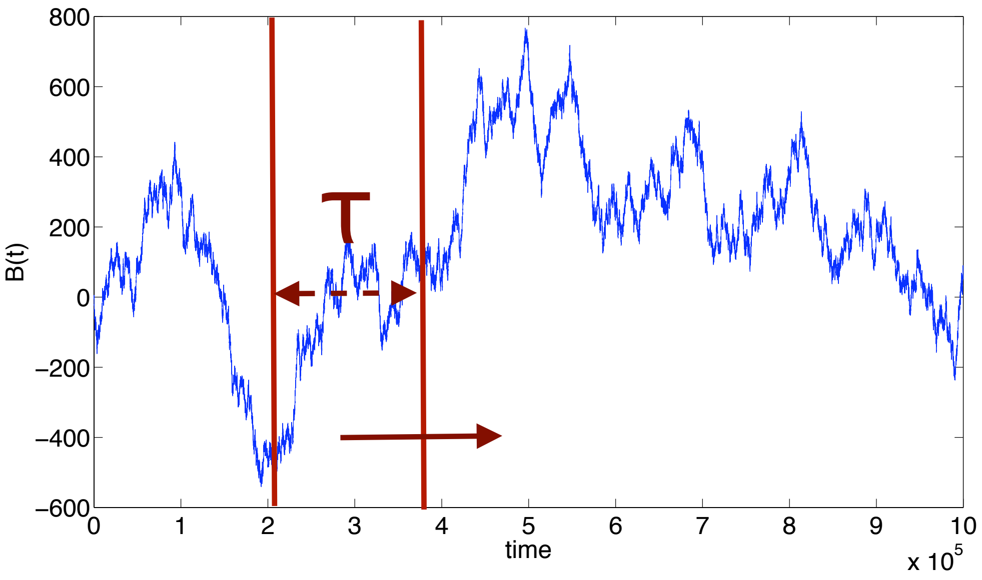


Change scale from t to bt
AND
scale x to $b^H x$

If the statistics of $b^H x$ is the same as x then process is statistically self-similar

Hurst exponent H

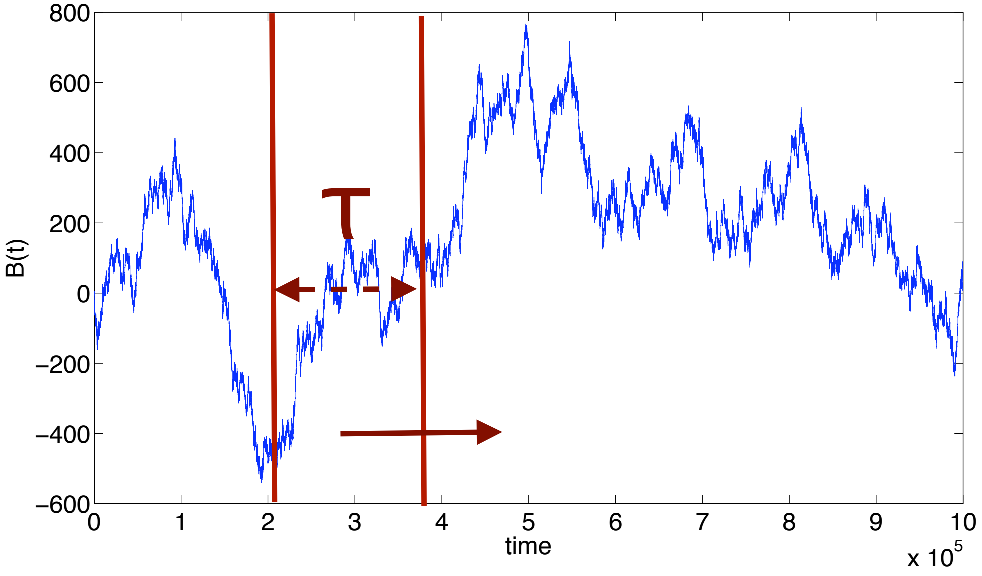
self-similarity



increments

$$y(t,\tau) = x(t+\tau) - x(t)$$

self-similarity



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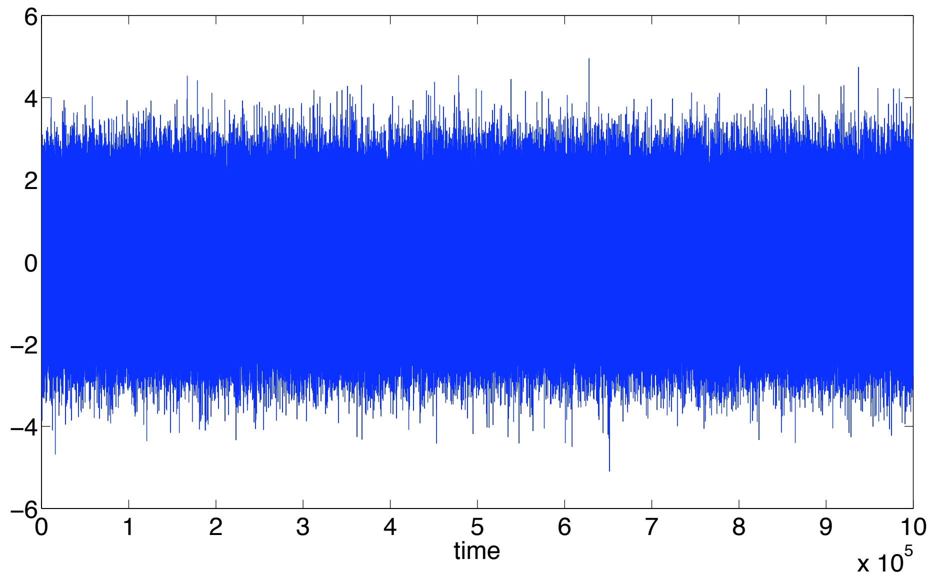
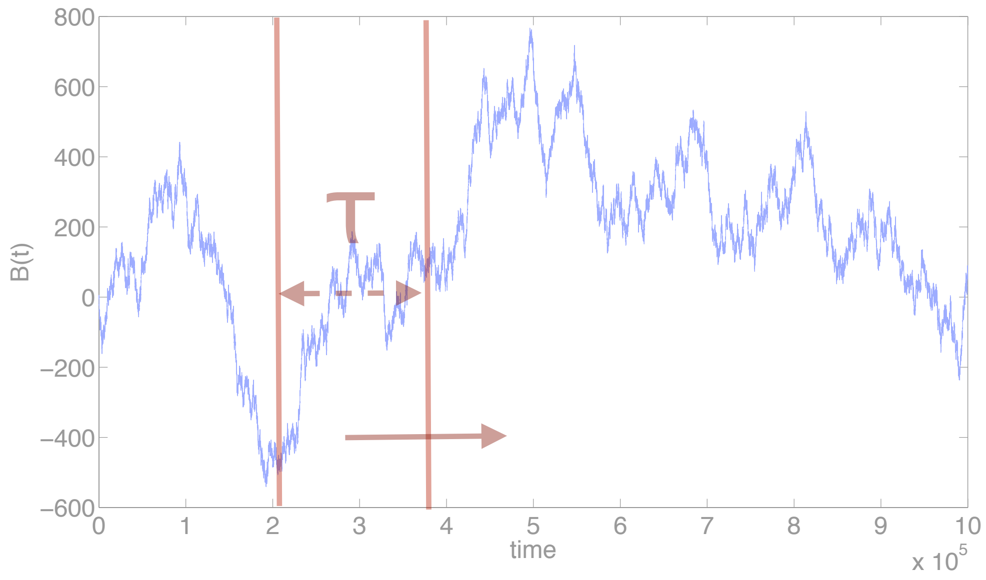
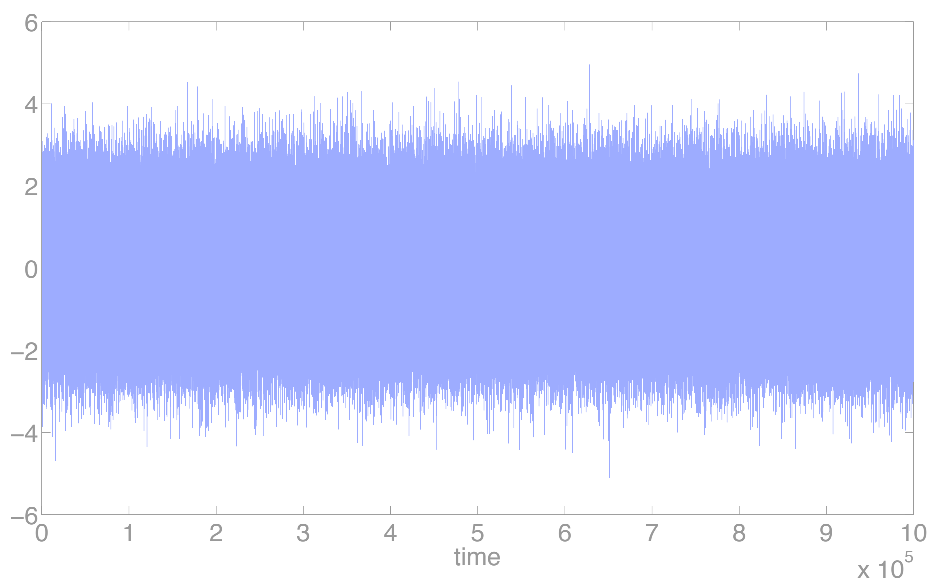
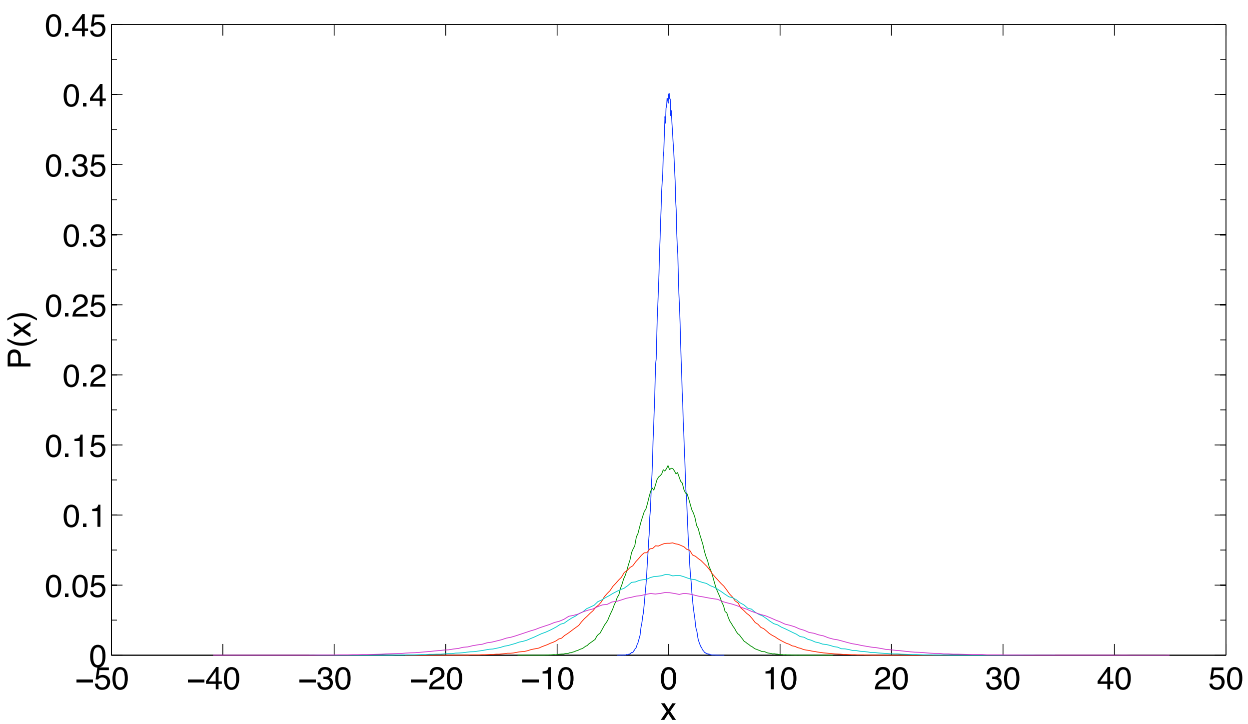


self-similarity



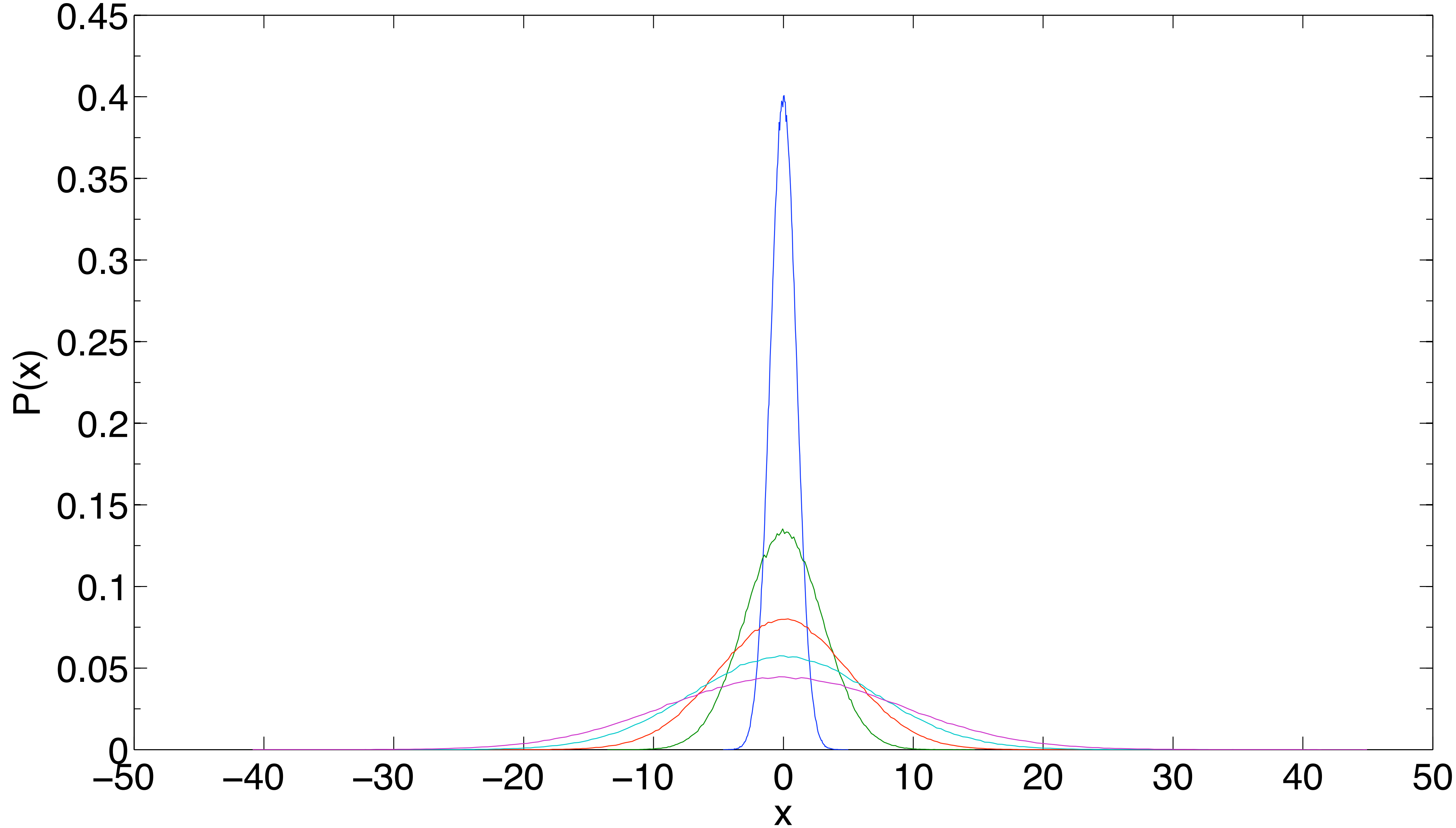
probability density function





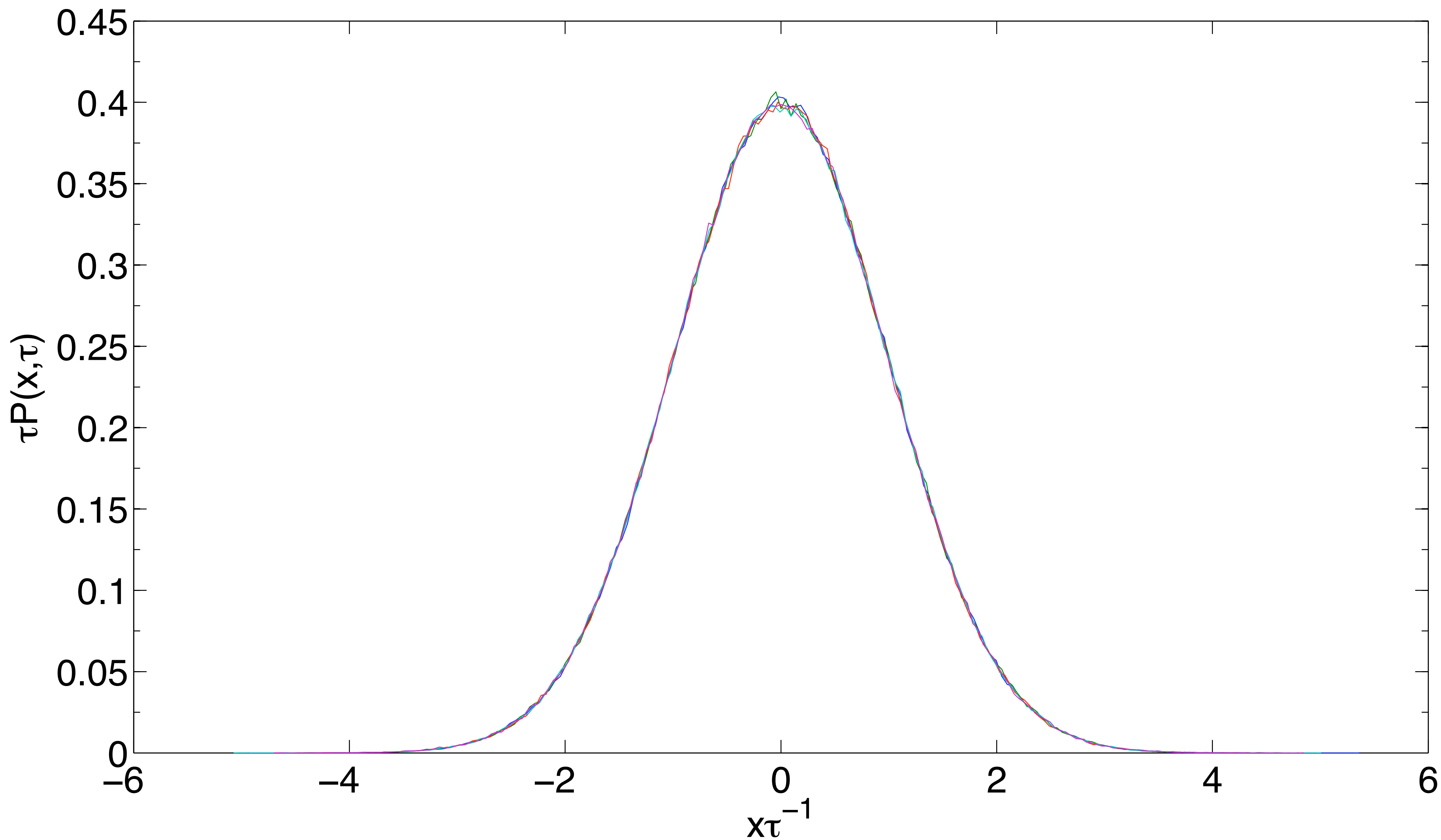
pdf scaling

$$P(y,\tau) = \tau^{-H}\mathcal{P}_s(y\tau^{-H})$$

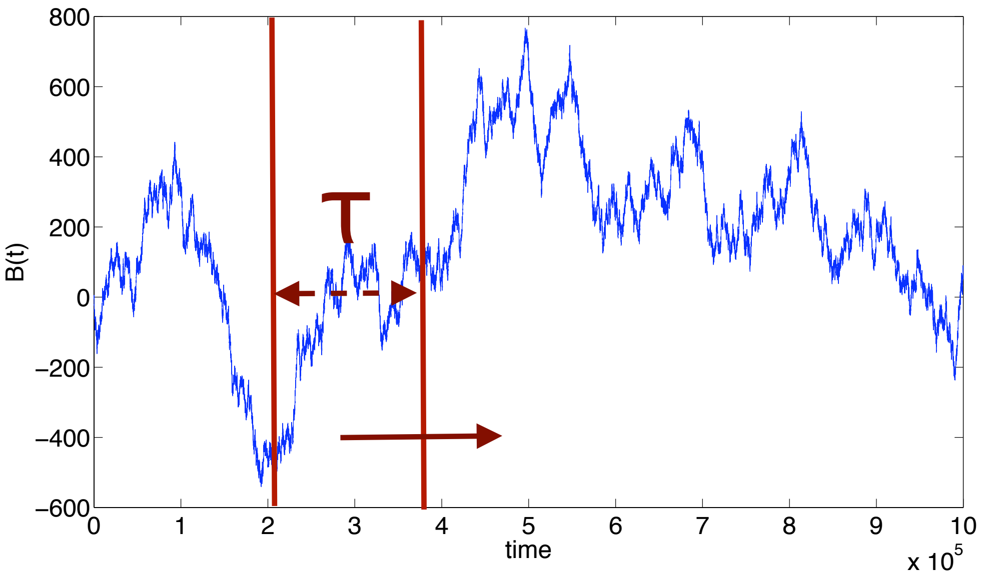


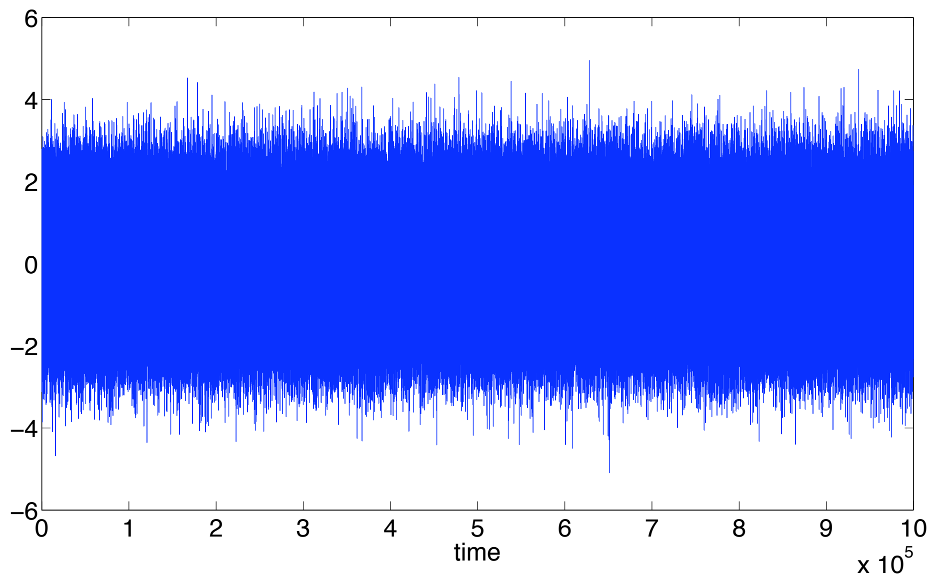
pdf scaling

$$P(y, \tau) = \tau^{-H} \mathcal{P}_s(y\tau^{-H})$$



Test statistic and its scaling

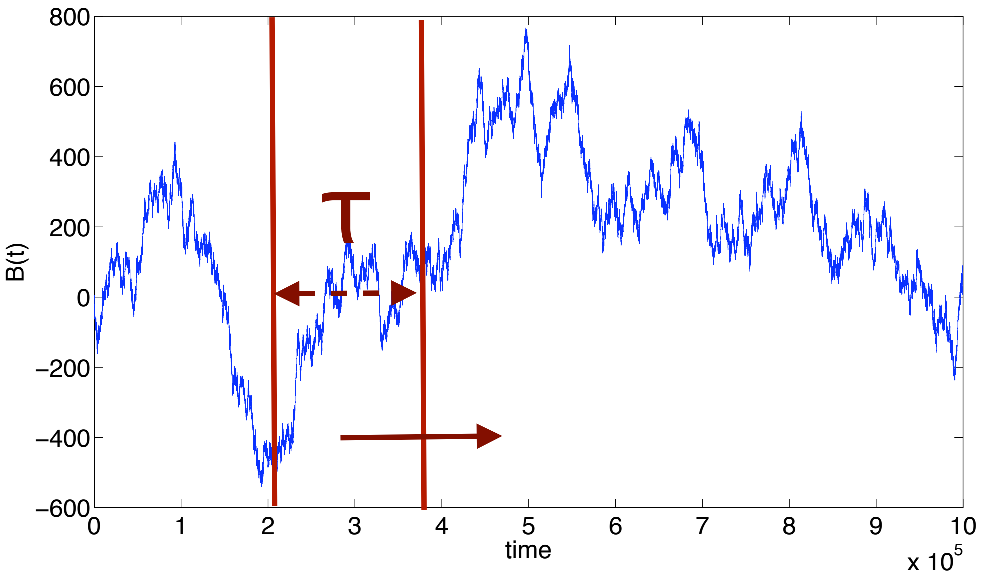


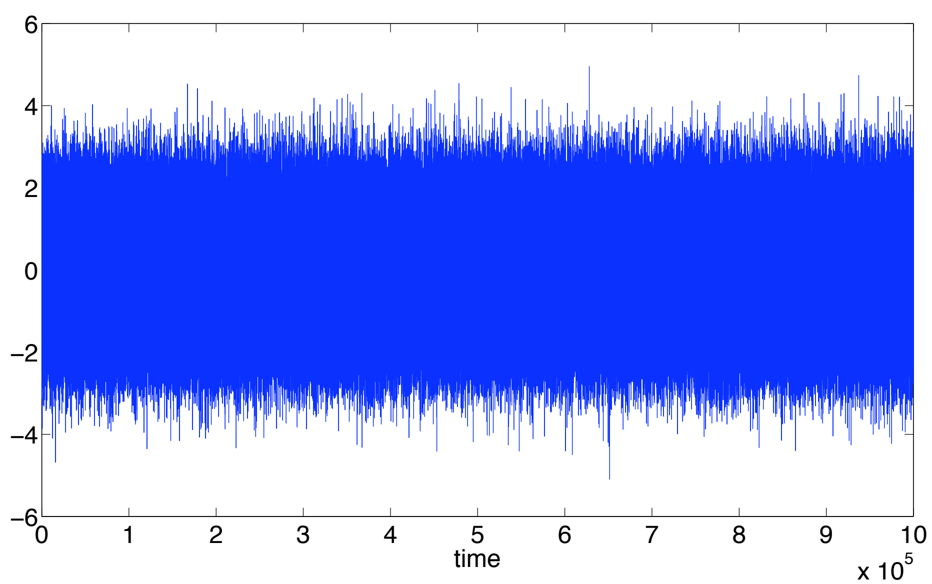


increments

$$y(t,\tau) = x(t+\tau) - x(t)$$

Test statistic and its scaling





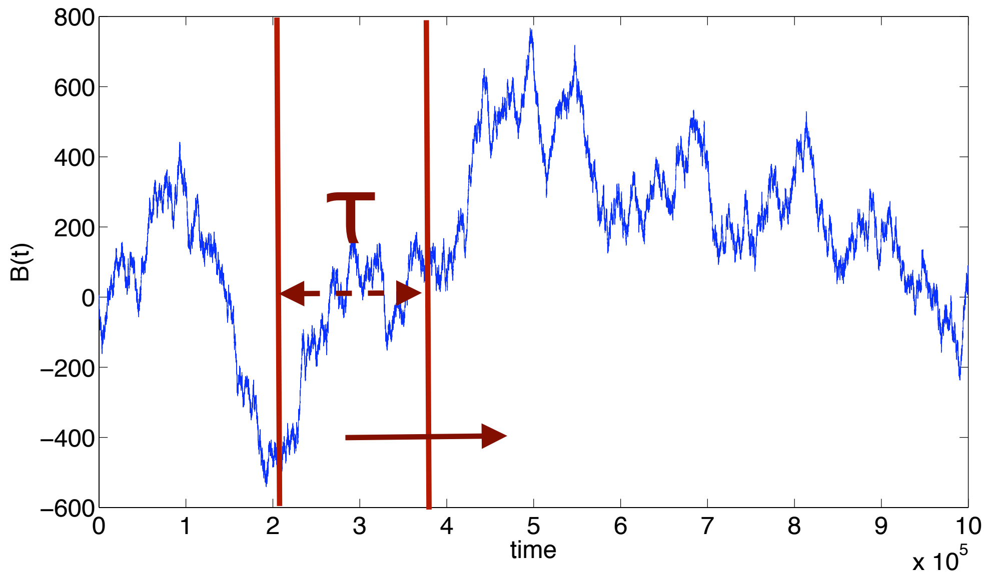
increments

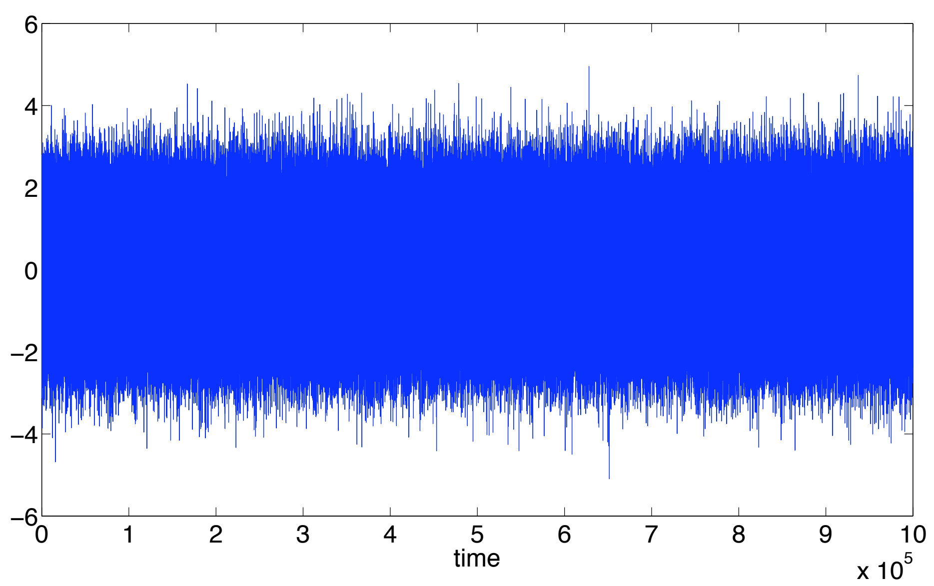
$$y(t,\tau) = x(t+\tau) - x(t)$$

p[th] order moment

$$M^p(\tau) = \frac{1}{N}\sum_{j=1}^{N} y_j^p$$

Test statistic and its scaling





increments

$$y(t,\tau) = x(t+\tau) - x(t)$$

p^th order moment

$$M^p(\tau) = \frac{1}{N}\sum_{j=1}^{N} y_j^p$$

moment scaling

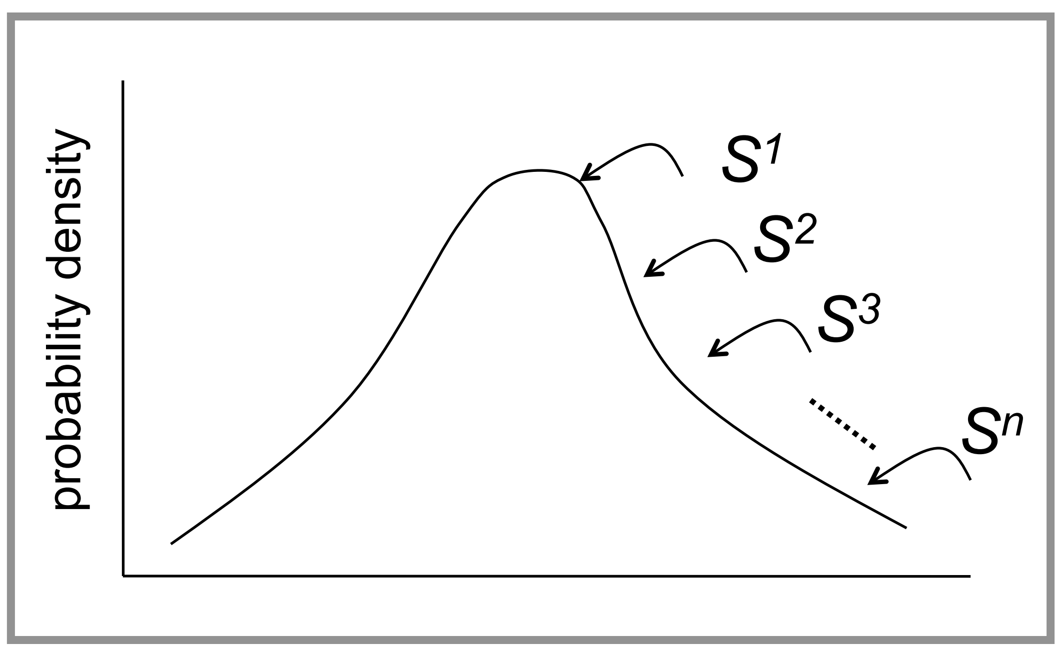
$$M^p(\tau) = M^p(1)\tau^{\zeta(p)}$$

$$\log M^p(\tau) = \log M^p(1) + \zeta(p) \log \tau$$

via ordinary least-squares regression

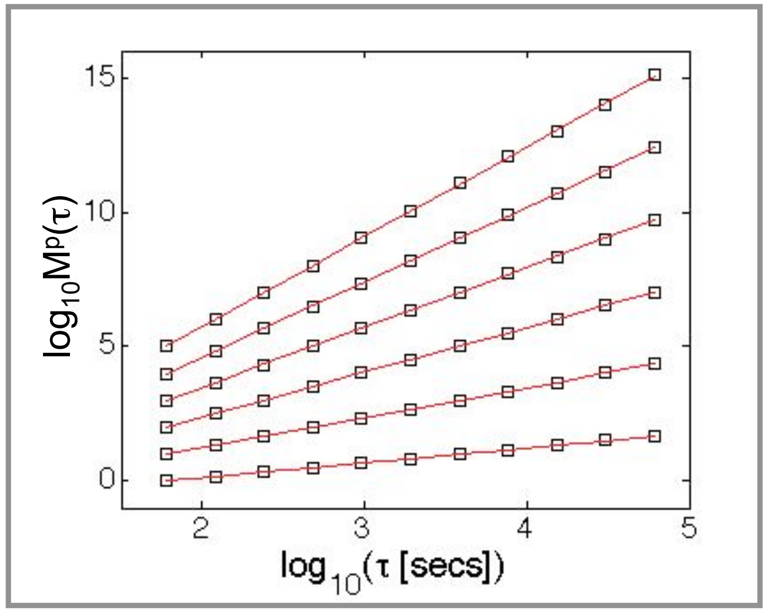
Heavy-tails and intermittency

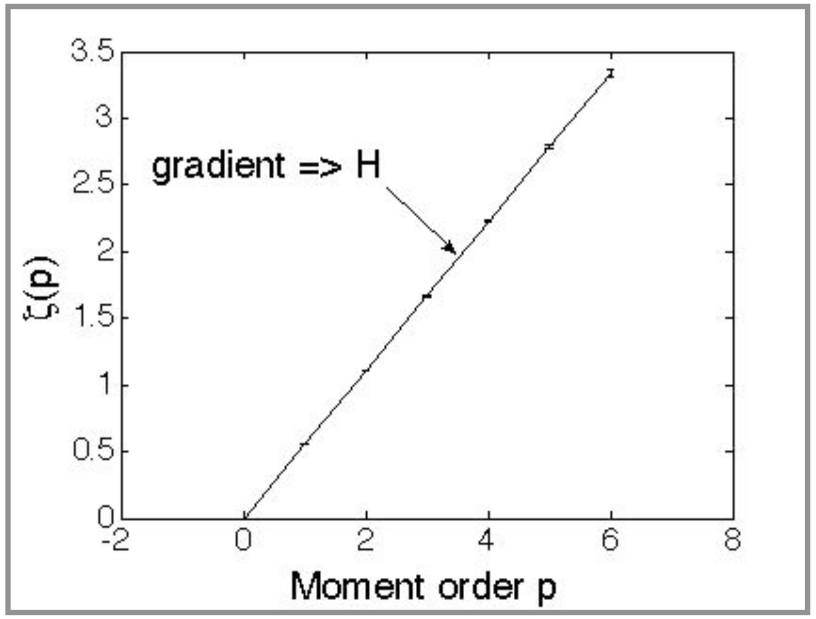
$$S^p(\tau) = \frac{1}{N}\sum_{i=1}^{N}|y_i^p|$$



Test statistic and its scaling

$$\log M^p(\tau) = \log M^p(1) + \zeta(p) \log \tau$$



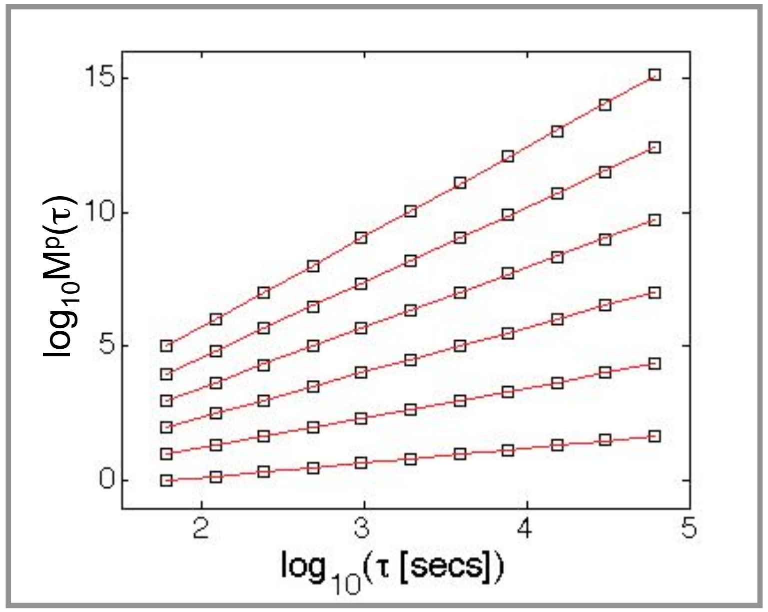


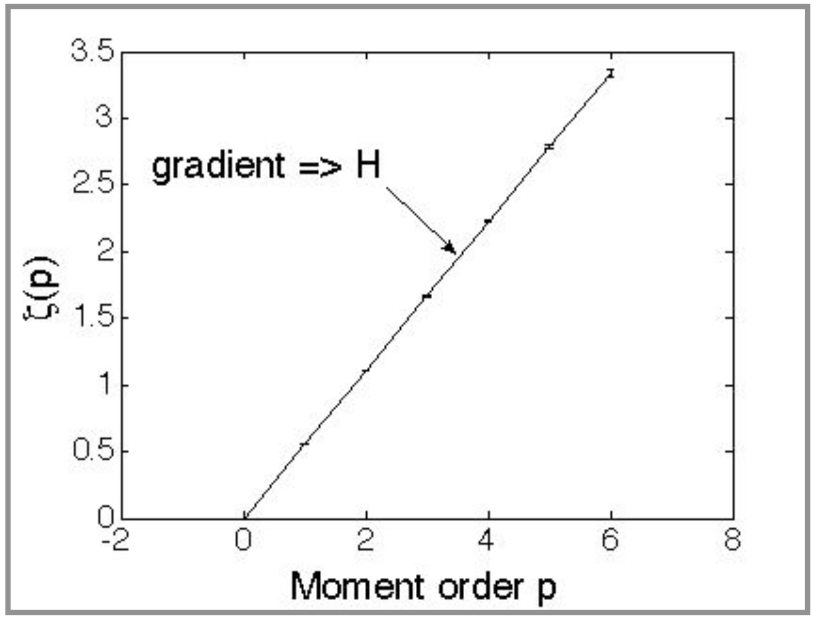
$$\zeta(p) = pH$$

single exponent scaling

Test statistic and its scaling

$$\log M^p(\tau) = \log M^p(1) + \zeta(p) \log \tau$$





$$\zeta(p) = pH$$

single exponent scaling

if $\zeta(p)\ non-linear$

then multi exponent scaling

Limit theorems and the origin of scaling

'All epistemological value of the theory of probability is based on this: that large scale random phenomena in their collective action create strict, non-random regularity.'

(Gnedenko and Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*)

Limit theorems

<u>Central Limit Theorem</u> (De Moivre, Laplace, Lyapunov)

$$S_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} y_i \qquad \lim_{N \to \infty} S_N \to Gaussian$$

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and many others! ⟶ Stable processes

Limit theorems and self-similar processes

$$S_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} y_i$$

Coarse-graining/averaging

Scaling

$$S_N = \frac{1}{N^{1/\alpha}} \sum_{i=1}^{N} y_i$$

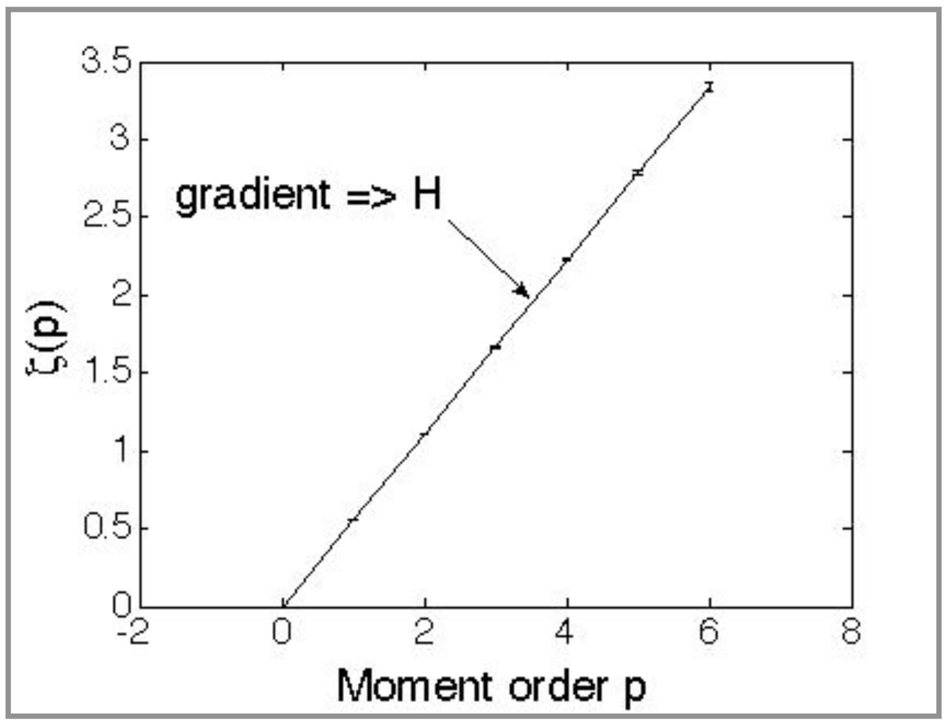
study of limit theorems and stable processes have a very profound link to <u>self-similar processes</u>

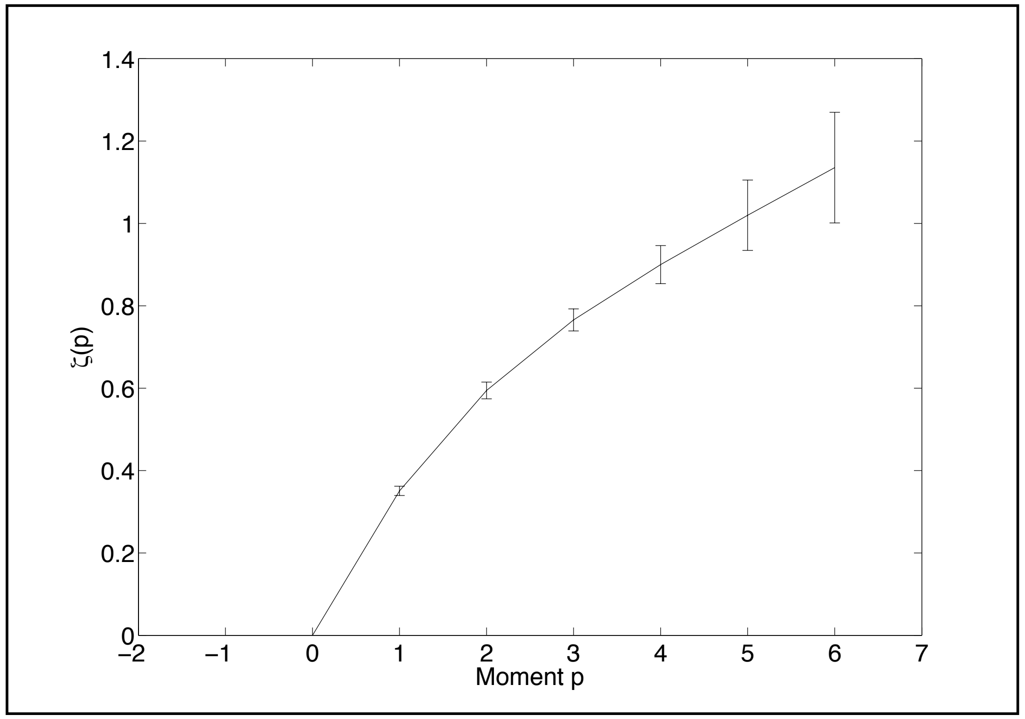
outline

- Equations of motion and phenomenology of energy transfer in r and k-space for iHI turbulence.
- Richardson energy cascade and the 5/3rd energy spectrum (power spectral density)
- Measurement and ensembles
- Higher order two-point statistics
- Some real data from the solar wind
- 4/5th (third order) law
- fractal models (if time permits)

Test statistic and its scaling

$$\log M^p(\tau) = \log M^p(1) + \zeta(p) \log \tau$$





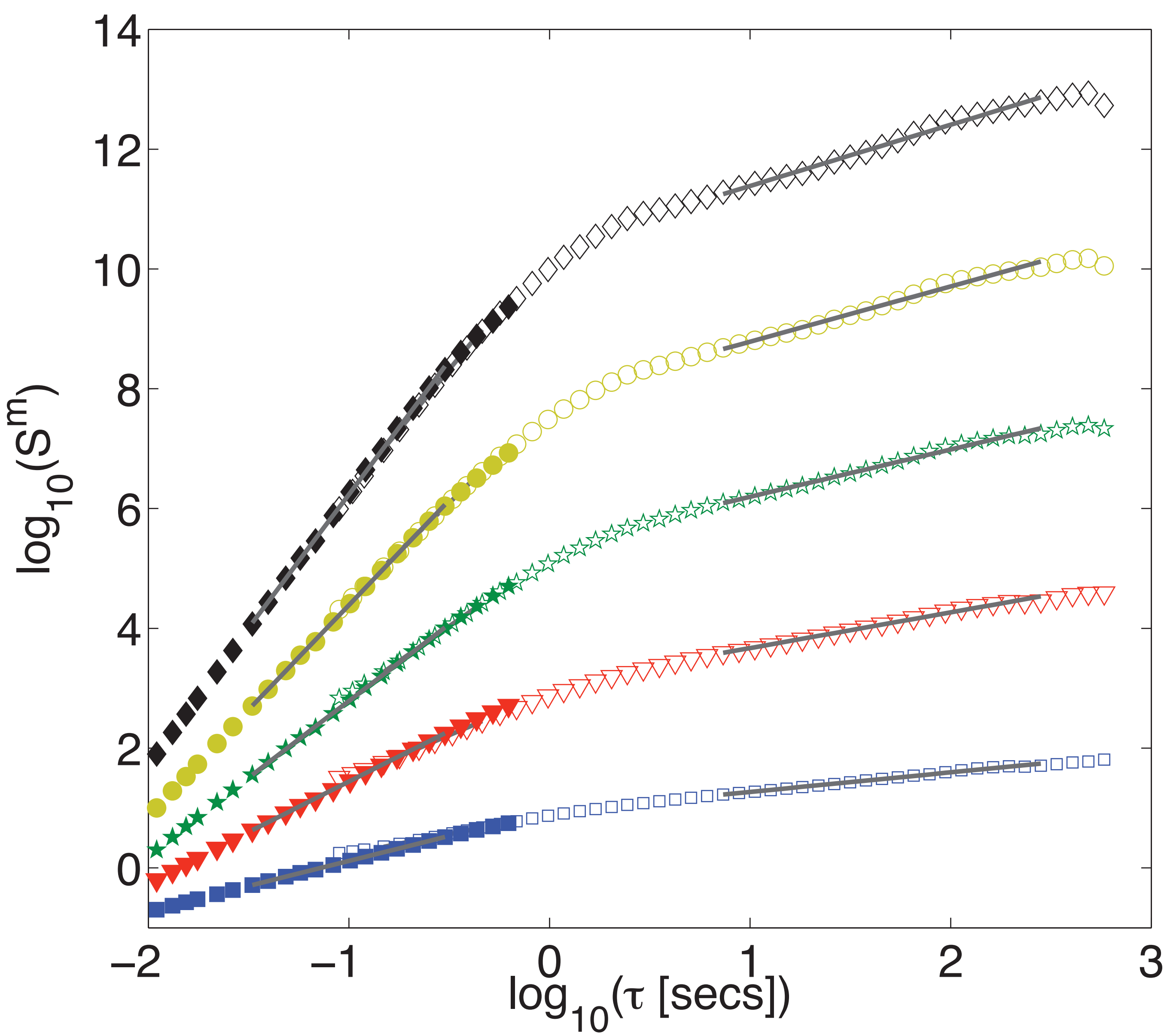
$$\zeta(p) = pH$$

monoscaling

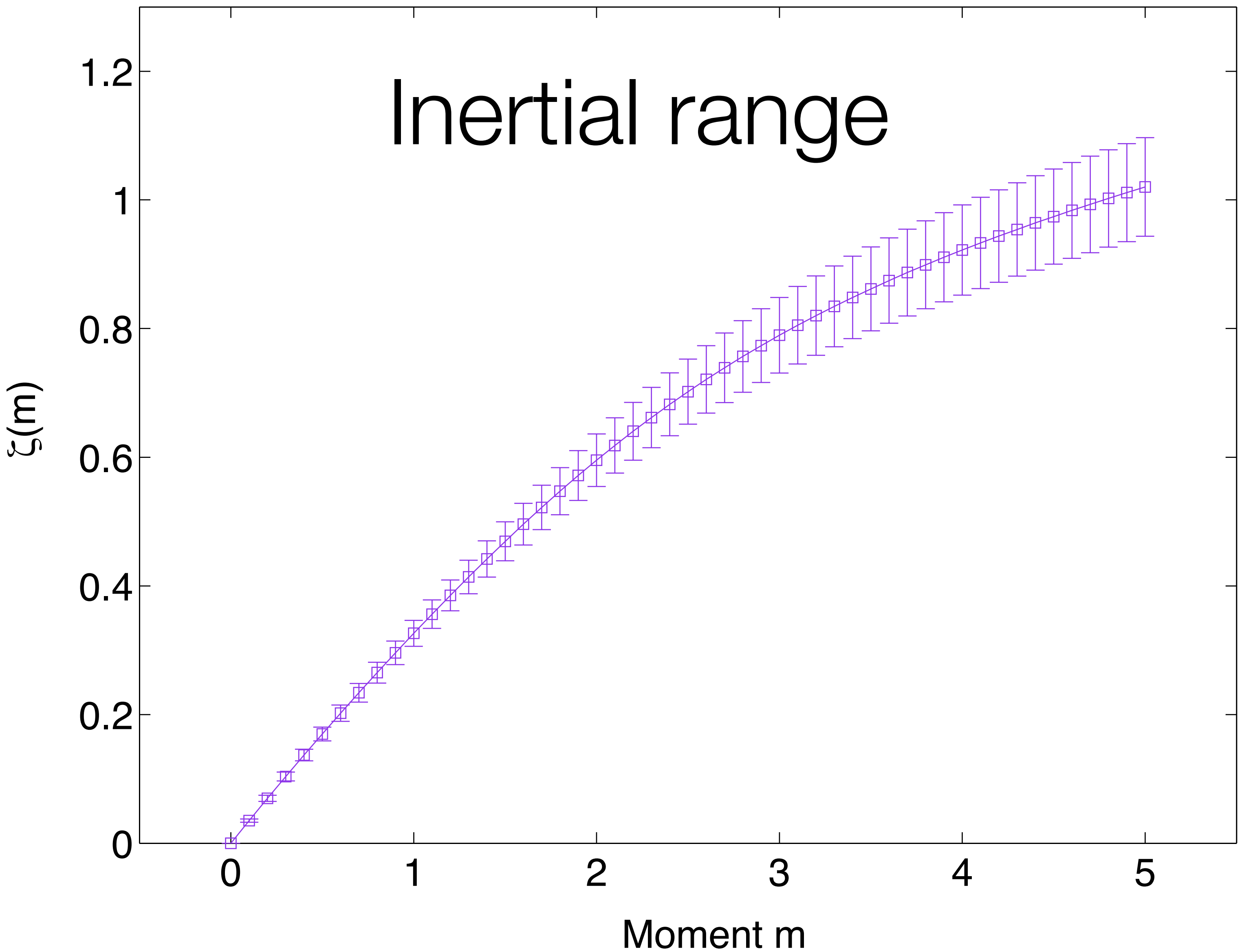
$$\zeta(p)\ non-linear$$

multiscaling

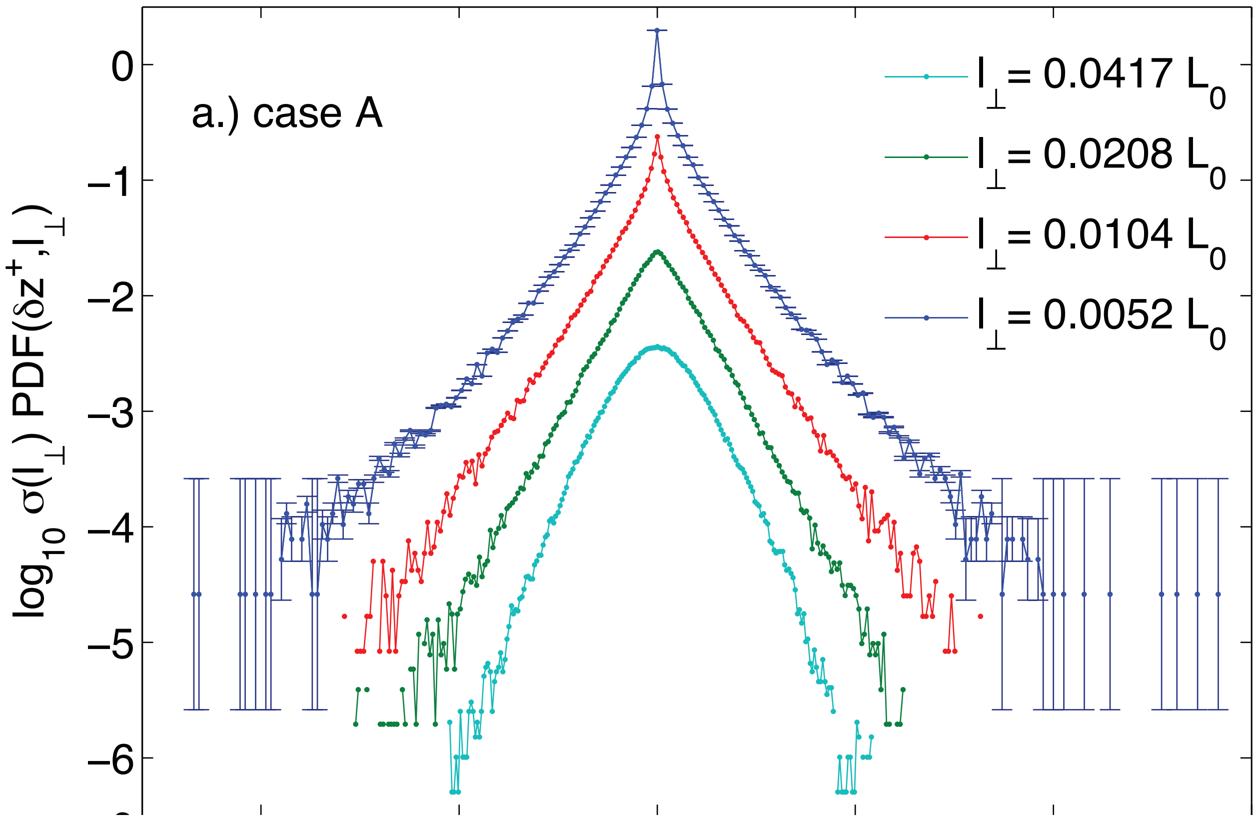
moment scaling



moment scaling



non-Gaussian pdfs



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4/5ths law (Kolmogorov 1941)

$$\frac{2}{3}\frac{\partial E}{\partial t} = -\frac{2}{3}\varepsilon = \frac{1}{2}\frac{\partial S_2}{\partial t} + \frac{1}{6r^4}\frac{\partial}{\partial r}(r^4 S_3) - \frac{\nu}{r^4}\frac{\partial}{\partial r}\left(r^4 \frac{\partial S_2}{\partial r}\right)$$

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$$S_3(r) \simeq -\frac{4}{5}\varepsilon r$$

why is the 4/5th law important?

- One of the few exact results from Navier Stokes Equations -- de facto **exact** closure of Karman Howarth equations
- Make a **direct measurement of the energy transfer rate** from simple structure functions i.e. ‘straight-forward’ **moment calculations**
- Energy in = Energy out => direct **measurement of total energy going into dissipation and heating** (thermodynamics of the system)
- Can be shown valid for each realisation – **not dependent on an ensemble average**
- Free from intermittency corrections

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what does all of this mean and why does it matter?

Power-laws, exponents -- why should we care?

- Many theories and models
  - main prediction is scaling behaviour and thus scaling exponents
    - directly measurable from observations

- Information on the scaling of statistical quantities and thus help in prediction of bulk properties of turbulent flows e.g. ability to calculate Reynolds stresses in Reynolds averaged equations (RANS).

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However

- Need to take a sober attitude to such things -- avoid the temptation to just fit straight lines to log-log plots. Statistics and errors need to be handled well.

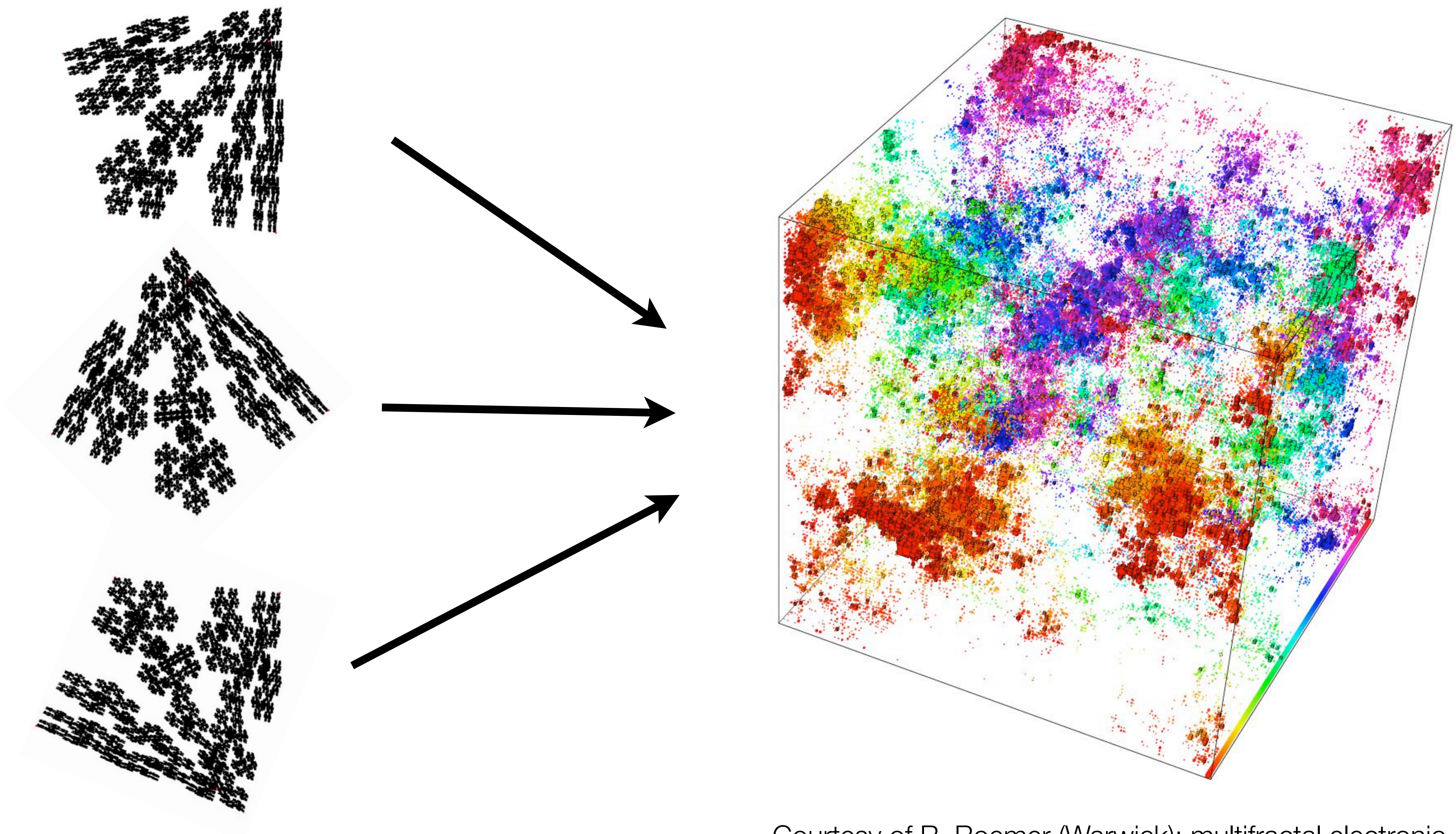
fractal dissipation

Single exponent *H* living on a fractal set of dimension *D* where dissipation occurs

Global Scale Invariance

Cantor 'dust' courtesy of Andrew Top: http://www.andrewtop.com/IFS3d/IFS3d.html

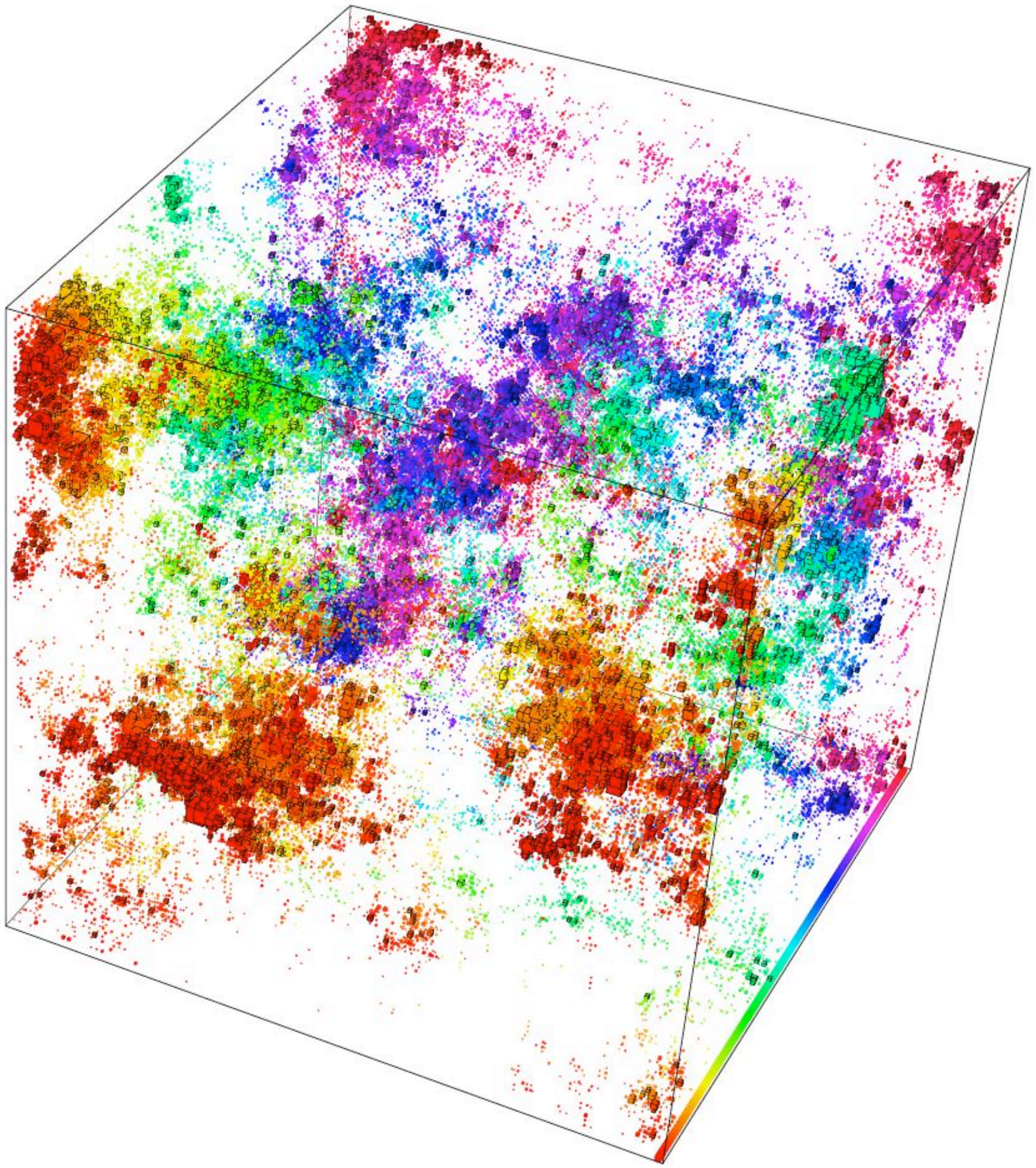
Multifractal dissipation



Courtesy of R. Roemer (Warwick): multifractal electronic wavefunction at metal insulator transition in 3D Anderson model

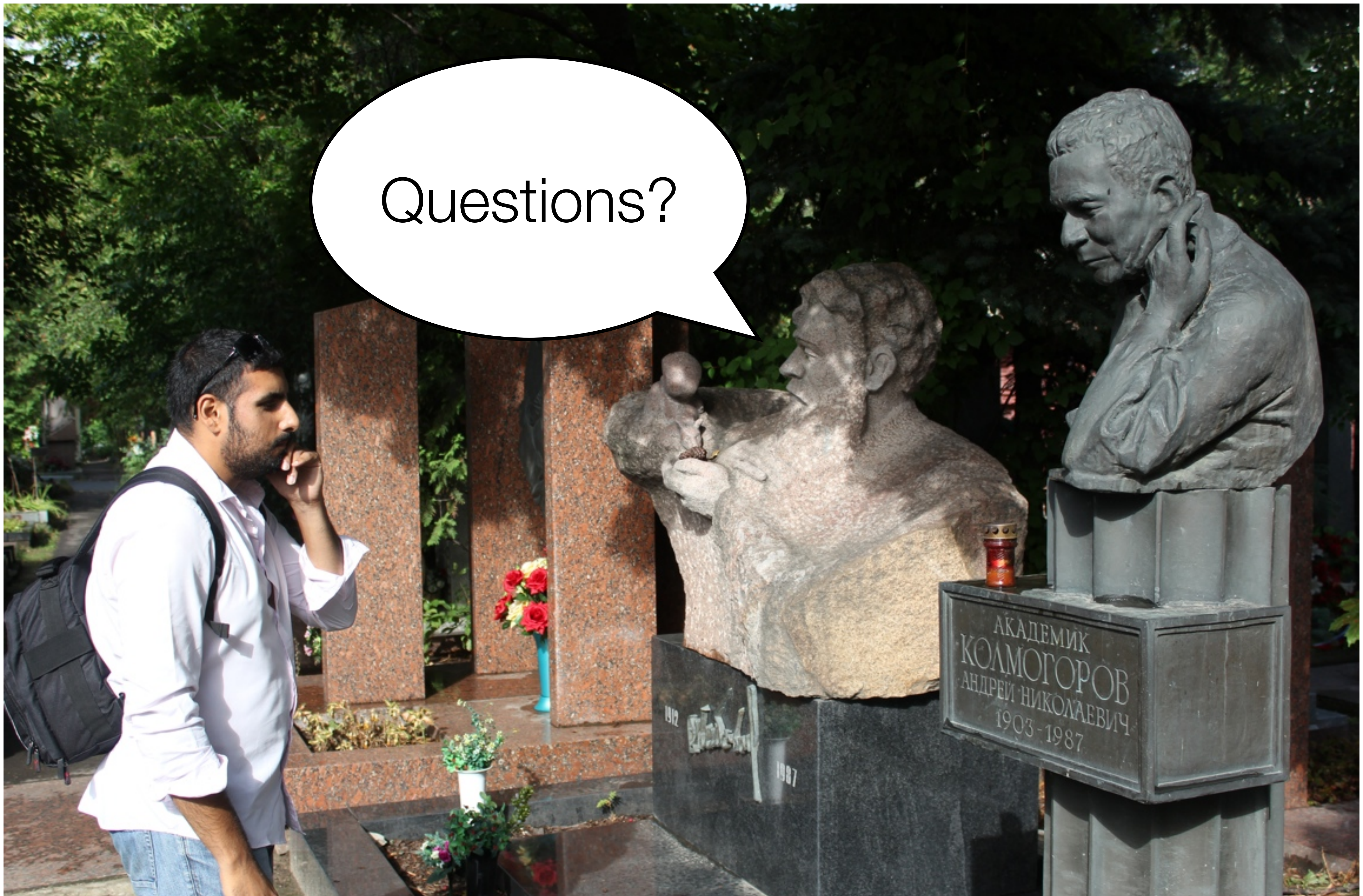
Multifractal dissipation

multiple exponents *h* living on a fractal sets of dimension *D(h)* where dissipation occurs



Local Scale Invariance

the end


Questions?
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КОЛМОГОРОВ
АНДРЕЙ НИКОЛАЕВИЧ
1903-1987
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