On Power-law Distributions of Observed Solar Features

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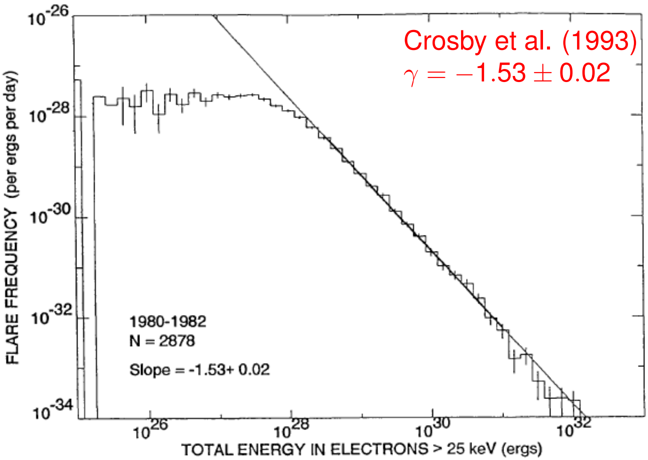
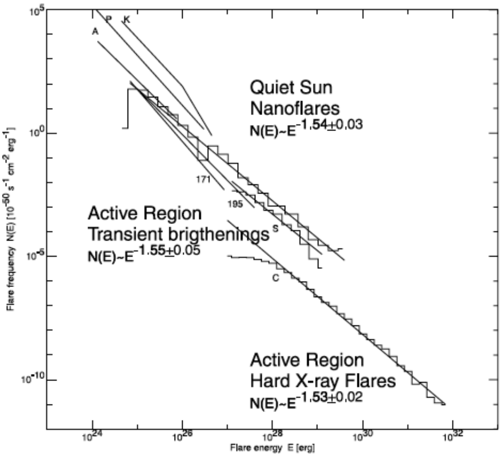
Outline

Outline

Distributions of Flare Energies



Distributions of Flare Energies



Aschwanden & Parnell (2002)
$\gamma = -1.54 \pm 0.03$

- K - Krucker & Benz (1998), Benz & Krucker (2002)
- P - Parnell & Jupp (2000) (corrected)
- A - Aschwanden et al. (2000)
- S - Shimizu (1995)
- C - Crosby et al. (1993)

Power-law indices of Nano/Microflares

Ref.	γ
Krucker and Benz 1998	2.59
Berghmans, Clette and Moses 1998	1.35
Parnell and Jupp 2000	2.52
Parnell and Jupp 2000	2.04
Aschwanden et al. 2000	1.80
Pres, Phillips and Falewicz 2001	2.47
Benz and Krucker 2001	2.31

Power law indices of nano-/micro-flare energy distributions (Benz & Krucker, 2001)

Outline

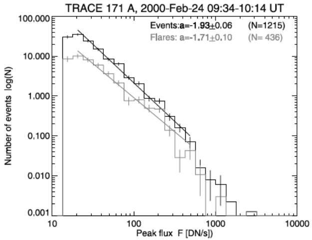
Power-law indices: Observational factors

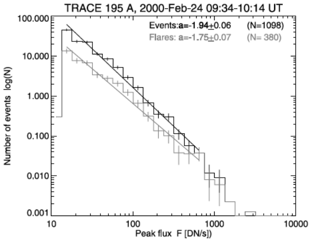
Ref.	γ	Method	Synch (mins)	Height model	Instru	Emission line	Select^{n}
KB 98	2.59	em	2	*const*	EIT	Fe IX/XII	no
BCM 98	1.35	rl	event	*none*	EIT	Fe XII	no
PJ 00	2.52	em	2	*const*	TRACE	Fe IX/XII	no
PJ 00	2.04	em	2	$A^{1/2}$	TRACE	Fe IX/XII	no
AEA 00	1.80	em	6	$A^{1/2}$	TRACE	Fe IX/XII	yes
PPF 01	2.47	em	?	?	Yohkoh	continuum	no
BK 01	2.31	em	2	$A^{1/2}$	EIT	Fe IX/XII	no

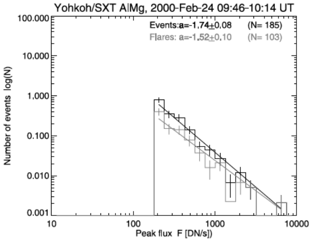
Power law indices of nano-/micro-flare energy distributions (Benz & Krucker, 01)

- Krucker and Benz (KB), Berghmans, Clette and Moses (BCM), Parnell and Jupp (PJ), Aschwanden et al. (AEA), Pres, Phillips and Falewicz (PPF) and Benz and Krucker (BK)
- Method: emission measure increase (em) or radiative loss (rl)
- Synchrony - time window used to determine events
- Height model - for density variation
- Selection - of flares

Comparison of Slopes: Observational Factors







Aschwanden & Parnell (2002)

Detections	γ
TRACE 171	
• Events (1215)	1.93 ± 0.06
• Flares (666)	1.71 ± 0.10
TRACE 195	
• Events (1098)	1.94 ± 0.06
• Flares (603)	1.75 ± 0.07
YOHKOH SXT	
• Events (185)	1.74 ± 0.08
• Flares (103)	1.52 ± 0.10

Outline

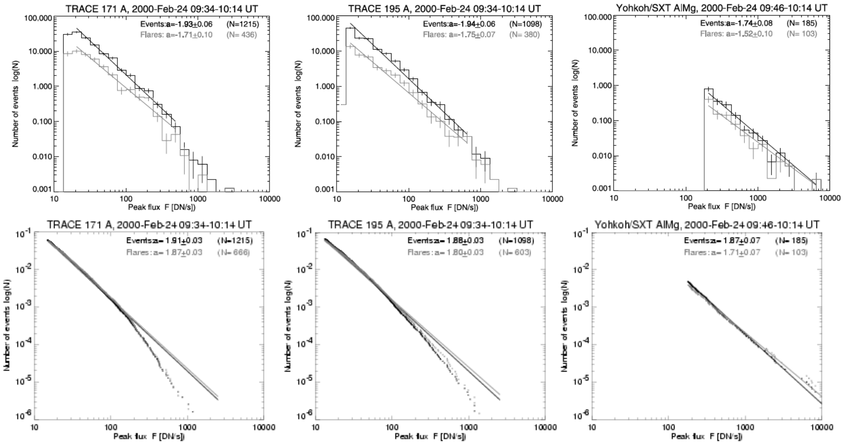
1. Observed Power Law Distributions
   - Distributions of Flare Energies
2. Factors that effect the power-law index
   - Observational factors
   - Fitting/Modelling distribution
3. Maximum Likelihood Estimates (MLE)
   - Method of MLE
   - Modelling a Power Law using MLE
   - Choosing range to fit power law to
   - Example & Plotting
   - Goodness of Fit
4. Summary

Power Law Fitting Methods

- Histogram: 4 choices
  - Bin size
  - Range of values to which power law fitted (cutoff)
  - Line fitting method
  - Weighted or unweighted
- Maximum Likelihood: 1 choice
  - Range of values to which power law fitted (cutoff)

$$\hat{\gamma} = \frac{1}{\mathrm{mean}(\log(E/\hat{E}_0))} + 1$$

Comparison of Slopes: Fitting Methods



Comparison of Slopes: Fitting Methods

Aschwanden & Parnell (2002)	Histogram line fit	Maximum Likelihood
TRACE 171		
• Events (1215)	1.93 ± 0.06	1.91 ± 0.03
• Flares (666)	1.71 ± 0.10	1.87 ± 0.03
TRACE 195		
• Events (1098)	1.94 ± 0.06	1.88 ± 0.03
• Flares (603)	1.75 ± 0.07	1.80 ± 0.03
YOHKOH SXT		
• Events (185)	1.74 ± 0.08	1.87 ± 0.07
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Histograms on small data numbers unreliable.

(See poster by Elke D'Huys)

Outline

Method of Maximum Likelihood Estimation

- Data set: E_i where $i = 1...n$
- Assume data are distributed with a particular form with probability density function: $f(E_i; \alpha_j)$, where α_j are $j = 1...m$ unknowns
- Find best estimates of parameters α_j:
  1. Define likelihood function: $l(\alpha_j) = \prod_{i=1}^{n} f(E_i; \alpha_j)$
  2. Take logs of likelihood function:
     $$L(\alpha_j) = \log(L(\alpha_j)) = \sum_{i=1}^{n} \log(f(E_i; \alpha_j)).$$
  3. Find *most likely* distribution by finding maximum of (log-)likelihood function by solving
     $$\frac{dL}{d\alpha_j} = 0$$

Outline

Modelling a Power Law Distribution using MLE

- Data set: E_i where $i = 1 \ldots n$
- Assume follows a power law distribution, so pdf:

$$f(E_i; \gamma, E_0) = \frac{\gamma-1}{E_0}\left(\frac{E_i}{E_0}\right)^{-\gamma}, \qquad E_i \geq E_0.$$

- Likelihood function: $l(\gamma, E_0) = \prod_{i=1}^{n} \frac{\gamma-1}{E_0}\left(\frac{E_i}{E_0}\right)^{-\gamma}$
- Log-likelihood function:

$$L(\gamma, E_0) = \sum_{i=1}^{n}\left[\log(\gamma-1) - \log(E_0) - \gamma \log\left(\frac{E_i}{E_0}\right)\right]$$

- Best estimate of parameters $\gamma = \hat{\gamma}$ and $E_0 = \hat{E}_0$:

$$\hat{\gamma} = 1 + \frac{n}{\sum_{i=1}^{n} \log\left(\frac{E_i}{E_0}\right)} \qquad \hat{E}_0 = \min(\mathrm{E_i})$$

Standard Errors on Estimated Parameters

- Calculating error on parameter $\hat{\alpha}_j$:

$$error_{\alpha_j} = \left(\frac{d^2 L}{d\alpha_j^2} \right)^{-1/2}$$

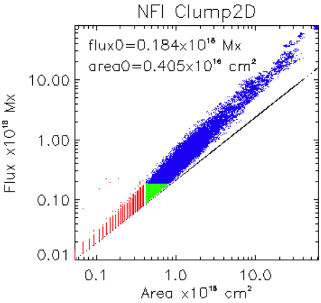
- Error for power law index
  - $error_\gamma = (\hat{\gamma} - 1) n^{-1/2}$

Outline

Identifying range to fit power law to (cutoff)

Need to identify a cutoff below which all energies ignored

- Observational constraints can cause under-estimation of events with given energies - important!



- Choose from looking at a plot of the data

Outline

Example data set

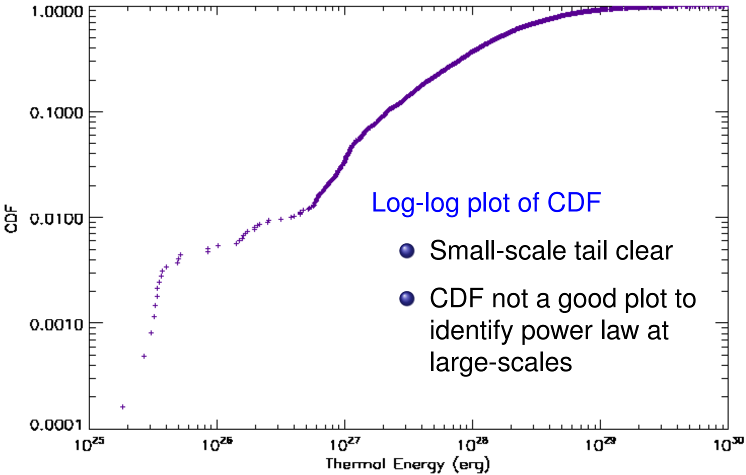
- Data from Hannah et al. (2008):
  - Energies of microflares (< C Class) from RHESSI between 2002-2007
  - Peak thermal energies (integrate over 16 secs about the time of peak 6-12keV emission)
  - Number of events: 3061
  - Range of flare energies: $1.70 \times 10^{25} - 1.36 \times 10^{30}$ ergs

Cumulative Distribution Functions (CDFs)

First look plots:

- CDF - gives the probability of having a value lower than E:
$$F(E) = \int_{E_0}^{E} f(x)dx$$
- Complementary CDF - gives the probability of having a value greater than E:
$$S(E) = 1 - F(E)$$
- To calculate CDF & cCDF of data:
    - Sort data such that: $E_i < E_{i+1} \;\; \forall \, E_i \quad i = 1...n$
    - Empirical CDF: $F_i = \frac{i-1/2}{n}$
    - Empirical cCDF: $S_i = 1 - F_i = \frac{n-(i-1/2)}{n}$
- Visualisation
    - CDF: Plot F_i against sorted E_i
    - cCDF: Plot $S_i = 1 - F_i$ against sorted E_i

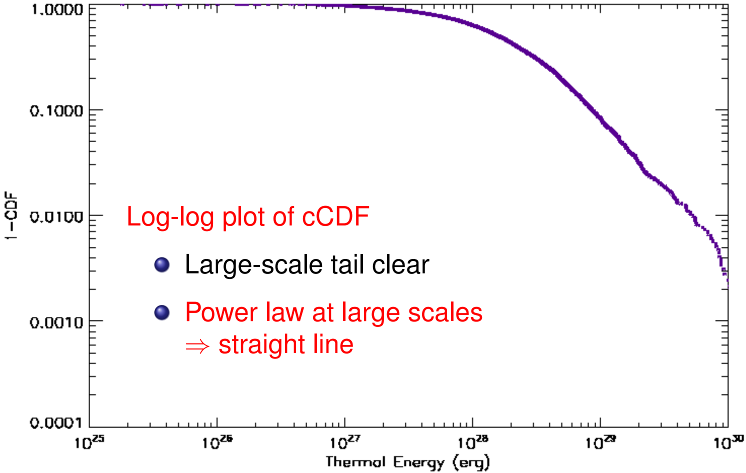
Cumulative Distribution Function (CDF)



Log-log plot of CDF

- Small-scale tail clear
- CDF not a good plot to identify power law at large-scales

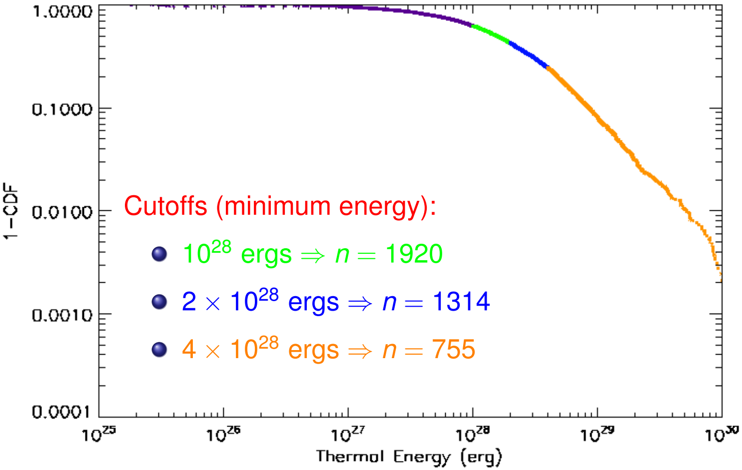
Complementary CDF (cCDF)



Log-log plot of cCDF

- Large-scale tail clear
- Power law at large scales
  $\Rightarrow$ straight line

Choose cutoff



Cutoffs (minimum energy):

- 10^{28} ergs $\Rightarrow n = 1920$
- 2×10^{28} ergs $\Rightarrow n = 1314$
- 4×10^{28} ergs $\Rightarrow n = 755$

Determine power law index using MLE

MLE Formulae for parameter estimations

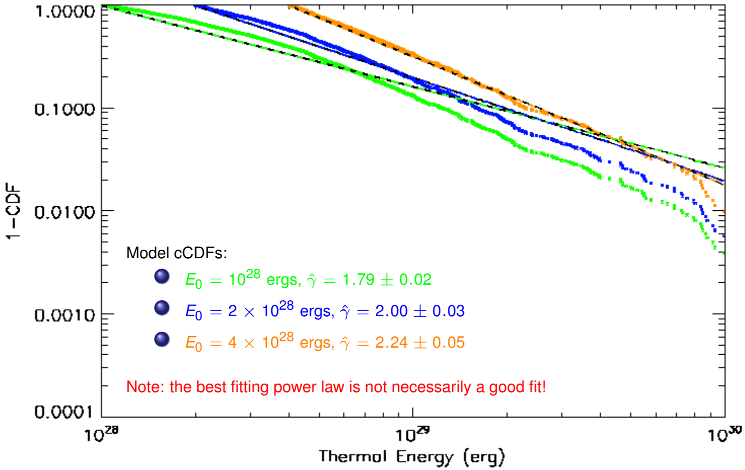
$$\hat{E}_0 = \min(E_i) \qquad \hat{\gamma} = 1 + \frac{n}{\sum_{i=1}^{n} \log\left(\frac{E_i}{\hat{E}_0}\right)}$$

- $n = 1920 \Rightarrow E_0 = 10^{28}$ ergs, $\hat{\gamma} = 1.79 \pm 0.02$
- $n = 1314 \Rightarrow E_0 = 2 \times 10^{28}$ ergs, $\hat{\gamma} = 2.00 \pm 0.03$
- $n = 755 \Rightarrow E_0 = 4 \times 10^{28}$ ergs, $\hat{\gamma} = 2.24 \pm 0.05$

Compare Empirical & Model cCDFs

- Empirical cCDF: $S_i = 1 - F_i = \frac{n-(i-1/2)}{n}$
- Power law cCDF: $1 - F(E_i; \hat{\gamma}, \hat{E}_0) = \left(\frac{E_i}{\hat{E}_0}\right)^{1-\hat{\gamma}}$
- Visualisation
  - Plot: $1 - F_i$ against sorted E_i
  - Over plot: $1 - F(E_i; \hat{\gamma}, \hat{E}_0)$ against E_i

Comparison of Empirical & Model cCDF

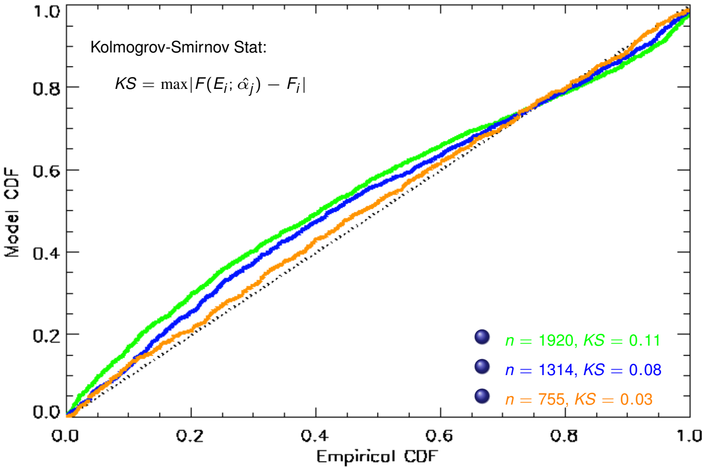


Outline

Absolute Goodness of Fit

- MLE determines the best pdf of given form...
  but this distribution may not be a good fit to the data!
- To determine if a pdf is a good fit to the data we make a percentile-percentile plot (P-P plot)
  - Plot Model CDF against Empirical CDF
    i.e., $F(E_i; \hat{\alpha}_j)$ against E_i

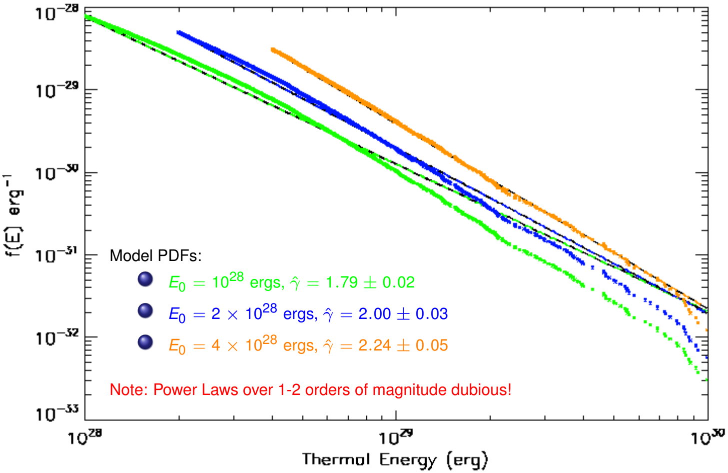
Example P-P plot



Compare Empirical & Model PDFs

- Power law PDF: $f(E_i; \hat{\gamma}, \hat{E}_0) = \frac{\hat{\gamma}-1}{\hat{E}_0}\left(\frac{E_i}{\hat{E}_0}\right)^{-\hat{\gamma}}$
- Empirical power law PDF: $f_i(\hat{\gamma}, \hat{E}_0) = \frac{\hat{\gamma}-1}{E_i}(1 - F_i)$
- Visualisation
  - Plot f_i against sorted E_i
  - Over plot $f(E_i, \hat{\gamma}, \hat{E}_0)$ against E_i

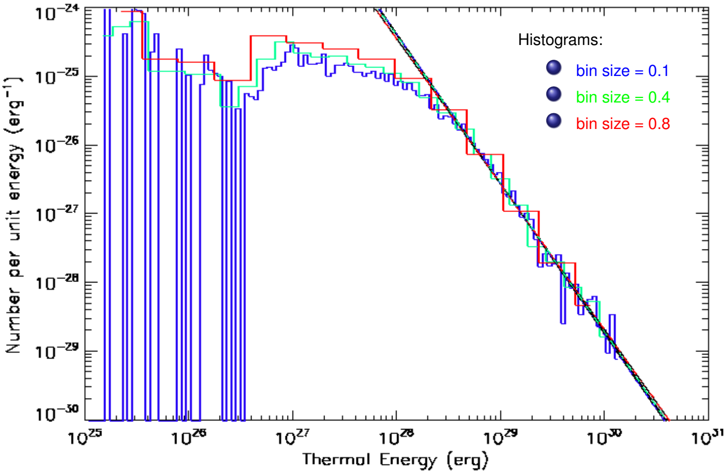
Empirical & Model PDFs



Slopes from Histogram Method?

E_{min} (ergs)	bin size	Number of bins	γ	Maximum Likelihood
10^{28}	0.1	49	1.98 ± 0.05	(1.79 ± 0.02)
10^{28}	0.4	12	2.00 ± 0.06	
10^{28}	0.8	6	1.98 ± 0.08	
2×10^{28}	0.1	42	2.10 ± 0.06	(2.00 ± 0.03)
2×10^{28}	0.4	10	2.13 ± 0.06	
2×10^{28}	0.8	5	2.10 ± 0.05	
4×10^{28}	0.1	36	2.18 ± 0.07	(2.24 ± 0.05)
4×10^{28}	0.4	9	2.16 ± 0.07	
4×10^{28}	0.8	4	2.12 ± 0.09	

Histogram Plot



Summary

- Maximum Likelihood Estimation:
  - Easy to use
  - Only one freedom: range over which power-law is fitted
  - Much more reliable for smaller data sets than creating a histogram (binning data)
- To be believable, your power law should cover several (>2) orders of magnitude